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W. C. Thacker

ABSTRACT. A transformation is developed, based upon the scale dependence of turbulent diffusion, that relates temporal and spatial spectra of turbulent kinetic energy. The basic idea is that an eddy diffusivity is appropriate when scales of the flow smaller than a length ℓ and a time t are unresolved. An expression similar to Heisenberg's for eddy diffusivity is used to obtain the connection between ℓ and t necessary to transform temporal spectra into spatial spectra. This transformation reveals a close connection between Webster's Site D spectrum of turbulent kinetic energy in the ocean and Okubo's diagrams of oceanic mixing. Furthermore, all spectra obtained from dimensional arguments satisfy this transformation.

1. Introduction

An important quantity in the theory of turbulence is $\boldsymbol{E}(\boldsymbol{k})$, the spatial spectrum of kinetic energy. To measure $\boldsymbol{E}(\boldsymbol{k})$ is difficult since it requires sampling the velocity field at many spatial points simultaneously. It is much easier to record a time series of the velocity at one point, from which $\Phi(\omega)$, the temporal spectrum of kinetic energy, can be obtained. Therefore a transformation is needed that will allow $\boldsymbol{E}(\boldsymbol{k})$ to be calculated if $\Phi(\omega)$ is known. The usual transformation is based upon Taylor's (1938) hypothesis of frozen turbulence, which is valid only if there is a strong mean flow. The purpose of this paper is to present a new transformation that should be valid in the absence of a mean flow. To stress the contrast with the idea of frozen turbulence the term "frost-free turbulence" is used.

The frost-free turbulence transformation is motivated by the results of dye-diffusion experiments in the ocean as summarized by Okubo's (1971) diagrams. His first diagram, reproduced here as Figure 1, illustrates that the spatial and temporal scales of turbulence can be related. This is certainly necessary if there is to be a transformation that can relate $\Phi(\omega)$ and $\boldsymbol{E}(\boldsymbol{k})$. His second diagram, Figure 2, shows the scale dependence of the eddy diffusivity. An expression for scale dependent diffusivity, such as Heisenberg's (1948) expression for eddy viscosity, is central to this transformation. The transformation is insensitive to the exact form of this expression because $\Phi(\omega)$ and $\boldsymbol{E}(\boldsymbol{k})$ fall off rapidly with increasing ω and $\boldsymbol{k}$.

This transformation is obtained in two ways. First $\Phi(\omega)$ and $\boldsymbol{E}(\boldsymbol{k})$ are related through the more general spectral density, $\boldsymbol{S}(\boldsymbol{k},\omega)$, which expresses both spatial and temporal variations. A comparison is made with the frozen turbulence case, and a more general transformation is suggested that has the limits of frozen turbulence and frost-free turbulence, depending upon whether advection or diffusion dominates. Then a heuristic derivation is given, based upon a mechanism for turbulent mixing. The idea here is that the mixing is due to shear dispersion on all scales. The results of a two-layer model for the shear effect are iterated over all scales to obtain expressions for the scale dependence of turbulent diffusivity from which the frost-free turbulence transformation follows.

Because it is difficult to measure $\boldsymbol{E}(\boldsymbol{k})$, it is difficult to test the validity of this transformation directly. Two indirect tests are discussed here. The first is a comparison of a temporal spectrum of kinetic energy of turbulence in the ocean obtained by Webster (1969) with the diffusion data displayed by Okubo (1971). It should be emphasized that no theory is presented for the observed shape of this spectrum. That is a dynamical problem, and the transformation discussed here should be regarded as kinematical. The comparison shows that the two types of data are in excellent agreement. The second test is provided by a dimensional argument. If it is assumed that only one dimensional constant is important, then consistent forms for $E(k)$ and $\Phi(\omega)$ can be obtained. Again, it should be emphasized that it is unimportant whether such a dimensional argument can be applied to real data. What is important is that the forms for $E(k)$ and $\Phi(\omega)$ obtained from the dimensional argument are indeed related through the frost-free turbulence transformation.

2. The Frost-Free Turbulence Transformation

A turbulent velocity field can be considered to be a random function of space and time. If the turbulence is statistically homogeneous, isotropic, and stationary, then the variance in the velocity field $\langle u^2(x,t)\rangle$ can be represented in terms of spectral density, $S(k,\omega)$. The temporal and spatial spectra are obtained from $S(k,\omega)$ by integrating over wavenumbers and frequencies, respectively:

$$\left.\begin{aligned} \Phi(\omega) &= \int_0^\infty dk\, S(k,\omega) \\ E(k) &= \int_0^\infty d\omega\, S(k,\omega) \end{aligned}\right\} . \qquad (1)$$

Thus, Φ is related to E through $S(k,\omega)$.

In general, a single frequency does not correspond to a single wavenumber. Nevertheless, it is clear that high frequencies correspond to high wavenumbers and low frequencies to low wavenumbers. For example, oceanic motion with a scale of hundreds of kilometers is expected to correspond to time variations on the scale of months, not seconds. Therefore, it should be reasonable to assume that, for any wavenumber, $S(k,\omega)$ is sharply peaked at a single frequency and, for any frequency, $S(k,\omega)$ is peaked at a single wavenumber. This can be expressed in two ways,

$$\left.\begin{aligned} S(k,\omega) &= E(k)\delta(\omega - f(k)) \\ S(k,\omega) &= \Phi(\omega)\delta(k - g(\omega)) \end{aligned}\right\} , \qquad (2)$$

which are equivalent if the functions f and g are the inverses of each other. The Dirac delta functions can be considered as approximating more general distributions with finite widths.

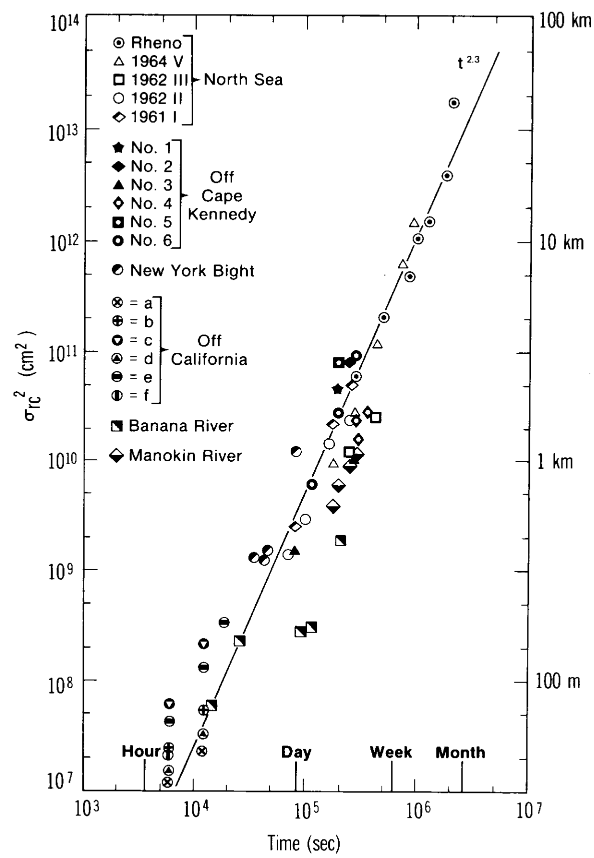

Figure 1. Variance of dye concentration (size of dye patch) versus diffusion time (time elapsed since the dye was introduced as a point source). (After Okubo, 1971.)

Using (1) with (2), transformations connecting E and Φ can be obtained:

$$\left.\begin{aligned} \Phi(\omega) &= \frac{E(g(\omega))}{f'(g(\omega))} \\ E(k) &= \frac{\Phi(f(k))}{g'(f(k))} \end{aligned}\right\} . \qquad (3)$$

Thus, the frost-free turbulence transformation will depend upon the form of the function f, its inverse g, and their derivatives f' and g'.

Clues for the form of f can be found in Okubo's (1971) dye diffusion diagrams. Figure 1 shows the relationship between the width of a dye patch and the duration of the dye diffusion experiment. This is the connection between space and time scales that is to be expressed by f. The fact that the slope of the line drawn through the data is greater than one indicates that turbulent diffusion

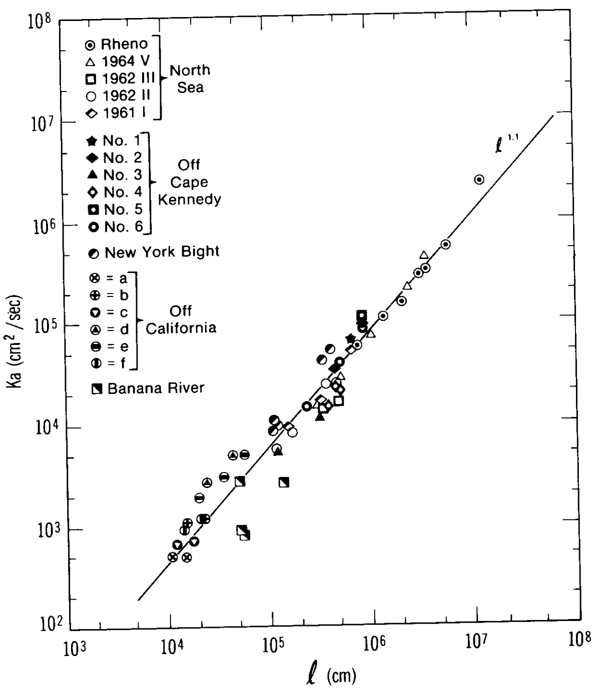

Figure 2. A diffusion diagram for apparent diffusivity versus scale of diffusion. (After Okubo, 1971.)

is scale dependent. Okubo obtains a scale dependent diffusivity K from these data using the expression

$$\ell^2 = 2Kt \quad (4)$$

to relate the width of the dye patch ℓ to the duration of the experiment t. Figure 2 shows K plotted against ℓ. Equation (4), plus an expression for the scale dependence of K, gives the relation between spatial and temporal scales necessary to define the functions f and g.

It is easy to understand why turbulent diffusion should be scale dependent. The spreading of the dye patch can be due only to those eddies that are smaller than the dye patch. Larger eddies serve only to advect and to distort the dye patch. At a later time when the dye patch is larger, larger eddies are available to contribute their energy to the mixing. Thus, the mixing proceeds faster as the dye patch gets larger.

Those eddies smaller than the dye patch, having wavenumbers greater than $\frac{2\pi}{\ell}$, are parameterized by the diffusivity K. As ℓ gets larger, so does K. Thus, K is a function of $k = \frac{2\pi}{\ell}$. The expression for the scale dependence used here is

$$K = \left[C \int_k^\infty \frac{E(k')dk'}{k'^2} \right]^{1/2}, \quad (5)$$

where C is a dimensionless constant of proportionality of order one. This is an expression quite similar to that used by Heisenberg (1948) for eddy viscosity, and is exactly that found by Tchen (1973, 1975) and Nakano (1972) from their dynamic theories of turbulence. A heuristic derivation of this expression, based upon the idea that the mechanism of turbulent mixing is shear dispersion on all scales, is given below.

The parameter K, evaluated according to (5), accounts for the effects of eddies with wavenumbers larger than $k = \frac{2\pi}{\ell}$ where ℓ is the width of the dye patch. The basic assumption made here is that these eddies correspond to frequencies greater than $\omega = \frac{2\pi}{t}$, where t is the duration of the dye experiment corresponding to the width ℓ. Thus, ω is related to k and K by equation (4),

$$\omega = \pi^{-1} K k^2. \quad (6)$$

Equation (6) expresses the relationship between ω and k that is necessary to transform $\Phi(\omega)$ into $E(k)$. This should be thought of as a statistical assumption for several reasons. First, since, in general, there is no one-to-one relationship between frequencies and wavenumbers, the relationship expressed by (6) must be statistical in the sense that it is a "most likely" relationship. Second, it is statistical since the parameter K is assumed to represent the average effect of the small scales. It is clear that (6) should be valid only when the small scales can be described by an eddy diffusivity. Finally, implicit in (6) is the idea of ergodicity: a spatial average, a temporal average, and an ensemble average should all be equivalent. A time series of length t determines $\Phi(\omega)$ for $\omega > \frac{2\pi}{t}$. Likewise, a spatial profile of length ℓ determines $E(k)$ for $k > \frac{2\pi}{\ell}$. If the time series is measured simultaneously with the dye experiment, it seems most reasonable to relate ω and k according to (6). Clearly, the results for each experiment should vary somewhat, but it is reasonable to think of a most likely result that represents the average of an ensemble of experiments. It is in this way that (6) should be interpreted.

By substituting (5) into (6), the expression for f is determined;

$$f(k) = \pi^{-1} k^2 \left[C \int_k^\infty \frac{E(k')}{k'^2} dk' \right]^{1/2}. \qquad (7)$$

The corresponding expression for g is found by inverting f. The simplest way to do this is to take advantage of the fact that $E(k)$ is simply a transformation of $\Phi(\omega)$. This implies that K can also be expressed as an integral of $\Phi(\omega)$ over frequencies greater than ω, and that expression can be used in (6) to obtain g. To obtain that expression, first write (5) in differential form as $dK = -C \frac{E(k)dk}{2Kk^2}$, and then use (6) and the identity $E(k)dk = \Phi(\omega)d\omega$ to obtain $dK = -\frac{C}{2\pi} \frac{\Phi(\omega)d\omega}{\omega}$. This can be integrated to give

$$K = \frac{C}{2\pi} \int_\omega^\infty \frac{\Phi(\omega')d\omega'}{\omega'} . \qquad (8)$$

Now, (8) and (6) yield

$$g(\omega) = \left[\frac{C}{2\pi^2 \omega} \int_\omega^\infty \frac{\Phi(\omega')d\omega'}{\omega'} \right]^{1/2}. \qquad (9)$$

Equations (7) and (9), together with (3), determine the frost-free turbulence transformation. It is simplest to write this in terms of K,

$$\left.\begin{aligned} \Phi(\omega) &= \frac{2\pi K E\left(\sqrt{\frac{\pi\omega'}{K}}\right)}{4kK^2 - C E\left(\sqrt{\frac{\pi\omega'}{K}}\right)} \\ E(k) &= \frac{4kK^2 \Phi(\pi^{-1}Kk^2)}{2\pi K + C\Phi(\pi^{-1}Kk^2)} \end{aligned}\right\}, \qquad (10)$$

where K is given by (5) and (8), respectively. Equations (5) and (8) will be discussed further in section 5. Equations (10) will be compared with results of experiments in section 3 and with results of dimensional arguments in section 4.

Equations (2) can be used to obtain the frozen turbulence transformation also. For that case, $f(k) = Uk$ and $g(\omega) = \omega/U$, where U is the mean velocity that advects the frozen turbulence. Using (3), the frozen turbulence transformation is given by

$$\left.\begin{aligned} \Phi(\omega) &= \frac{1}{U} E\left(\frac{\omega}{U}\right) \\ E(k) &= U\Phi(Uk) \end{aligned}\right\} .$$

It is possible to construct a more general transformation that reduces to frozen turbulence when advection dominates diffusion and to frost-free turbulence when diffusion dominates advection. One possibility is through the use of the function

$$f(k) = Uk + \pi^{-1} Kk^2$$

and its inverse, where K is given by (5) and (8), as before. This leads to the transformation

$$E(k) = \frac{(2\pi UK + 4kK^2)\Phi(Uk + \pi^{-1}Kk^2)}{2\pi K + C\Phi(Uk + \pi^{-1}Kk^2)} .$$

If the trend is removed from the time series from which Φ is to be obtained, then the information concerning the mean velocity U is lost so the transformation given in (9) should be used.

3. A Comparison With Data

A direct test of the frost-free turbulence transformation given by equations (10) is impossible since the data necessary to evaluate $E(k)$ are unavailable. Nevertheless, long time series of the velocity at one point in the ocean have been obtained, so $\Phi(\omega)$ is available. Such a spectrum from Site D (Webster, 1969) is shown in Figure 3. This can be compared with Okubo's (1971) dye diffusion diagrams to check whether the transformation is reasonable.

It is possible to evaluate the diffusivity from (8) numerically using the data in Figure 3. However, for simplicity, these data are approximated by the formula

$$\Phi(\omega) \sim \omega^{-4/3}. \qquad (11)$$

This does not imply that there is any dynamic significance to the exponent $-4/3$. This exponent was chosen simply to represent the gross behavior of the spectrum over the entire range of frequencies. The details of the tidal and inertial peaks in the spectrum should contribute, at most, shoulders to the curve of diffusivity versus frequency scale. Substituting (11) into equation (8) yields

$$K \sim \omega^{4/3}.$$

Using this in (6), with $\ell = \frac{2\pi}{k}$ and $t = \frac{2\pi}{\omega}$, gives a relation between spatial and temporal scales,

$$\ell^2 \sim t^{7/3}. \qquad (12)$$

This is exactly the behavior shown in Figure 1. Of course, the data in Figure 2 are well described by

$$D \sim \ell^{8/7}, \qquad (13)$$

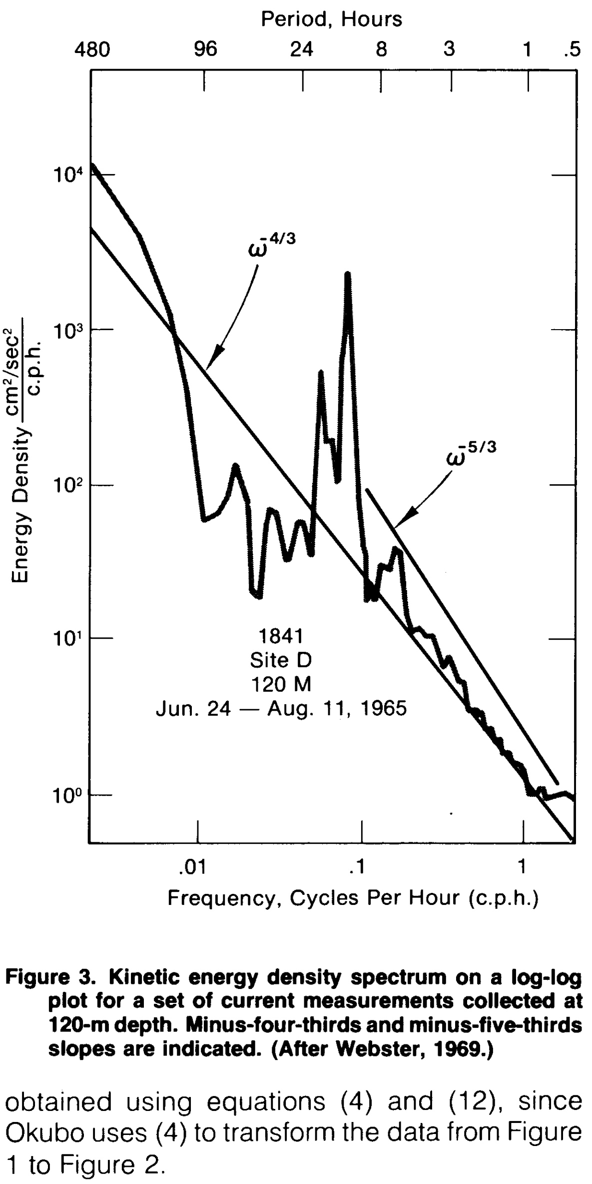

Figure 3. Kinetic energy density spectrum on a log-log plot for a set of current measurements collected at 120-m depth. Minus-four-thirds and minus-five-thirds slopes are indicated. (After Webster, 1969.)

obtained using equations (4) and (12), since Okubo uses (4) to transform the data from Figure 1 to Figure 2.

In making this comparison, two equations were used, (6) and (8). These are the two equations that define frost-free turbulence transformation. Equation (6) relates frequencies to wavenumbers through the eddy diffusivity, and equation (8) expresses the scale dependence of the eddy diffusivity. The agreement found when these two sets of data are compared in this way is evidence that this transformation is indeed valid.

4. A Comparison With A Dimensional Argument

A dimensional argument also provides a transformation connecting $\Phi(\omega)$ and $E(k)$. This argument is based upon the assumption that the spectra depend upon only one dimensional constant. For example, if this constant is ϵ, the rate of energy dissipation, then the dimensional argument gives the familiar results for the inertial subrange: $E(k) \sim k^{-5/3}$ and $\Phi(\omega) \sim \omega^{-2}$. No attempt is made here to argue that a single dimensional constant is appropriate for the entire spectrum shown in Figure 3. Perhaps it is possible to divide the spectrum into subranges in which a single constant is important, but that is not assumed. The point is that if such a subrange does exist, then the dimensional argument for the subrange is in agreement with the frost-free turbulence transformation.

Suppose that the one important dimensional constant Q has dimensions $X^a T^b$. Then E and Φ must have the following forms in order to be dimensionally correct:

$$\left.\begin{aligned} E(k) &\sim Q^{-\frac{2}{b}} k^{-\left(\frac{2a}{b}+3\right)} \\ \Phi(\omega) &\sim Q^{\frac{2}{a}} \omega^{\frac{2b}{a}+1} \end{aligned}\right\}. \tag{14}$$

For the case where Q is taken to be ϵ, the exponents a and b are 2 and -3, respectively.

These forms can be seen to be consistent with frost-free turbulence. Suppose that $\Phi(\omega) \sim \omega^{-p}$, where $p = -\left(\frac{2b}{a}+1\right)$, and calculate $E(k)$ using the frost-free transformation (10). The result is

$$E(k) \sim k^{\left(\frac{3p-1}{p+1}\right)}, \tag{15}$$

which is exactly what is given in (14) when

$$p = -\left(\frac{2b}{a}+1\right).$$

Dimensional arguments can also be made for the diffusivity and for the width of the dye patch:

$$\left.\begin{aligned} K &\sim Q^{-\frac{a}{b}} \ell^{2+\frac{a}{b}} \sim \ell^{\frac{2p}{p+1}} \\ \ell^2 &\sim Q^{\frac{2}{a}} t^{-\frac{2b}{a}} \sim t^{p+1} \end{aligned}\right\}. \tag{16}$$

These results are also in agreement with frost-free turbulence, as can be seen from equations (5) and (6) with $E(k)$ given by (15).

5. Shear Dispersion

It is possible to derive the frost-free turbulence transformation using the idea that the mechanism for turbulent mixing is shear dispersion on all scales. Such a derivation should clarify the idea of scale dependent diffusion that is intrinsic to the transformation described here. It should also clarify the manner in which the advection of small eddies by large eddies is incorporated into the transformation.

Shear dispersion was discussed by Taylor (1953, 1954) in the context of longitudinal dispersion in pipes. He found that an enhanced diffusivity was needed to account for the dispersion of contaminant introduced into the flow. This enhanced diffusivity is appropriate in conjunction with the cross-sectional average of the contaminant concentration. The enhancement is due to the combined action of shear and cross-shear mixing, features that are unresolved when we are dealing with cross-sectional averages, and whose effects are accounted for by the enhanced diffusivity.

This shear dispersion relates to turbulent diffusion in two ways. First, the eddy diffusivity can likewise be considered as parameterizing the details of the flow that are averaged out. The value of the diffusivity appropriate to a given scale is determined by those details of the flow that are smaller than this scale. Second, the mechanism of turbulent mixing is shear dispersion on all scales. At any scale there are eddies that provide shear, and there are smaller eddies that provide mixing across this shear. As the scale is increased, the eddy diffusivity must be enhanced to account for the additional shear and cross-shear mixing that is averaged out.

From the point of view of someone numerically modeling the flow, this is clear. Eddy diffusivity is used to parameterize mixing due to sub-grid scale motion. For a coarser numerical grid, a larger value of diffusivity is needed to account for the mixing that would be explicitly resolved on a finer grid.

A simple two-layer model of shear dispersion can show how the diffusivity is enhanced as the details of the shear are averaged over. This model is given by the equations,

$$\left.\begin{aligned} \frac{\partial C_1}{\partial t} + u_1 \frac{\partial C_1}{\partial x} &= -\frac{1}{T}(C_1 - C_2) + K\frac{\partial^2 C_1}{\partial x^2} \\ \text{and} \qquad \frac{\partial C_2}{\partial t} + u_2 \frac{\partial C_2}{\partial x} &= -\frac{1}{T}(C_2 - C_1) + K\frac{\partial^2 C_1}{\partial x^2} \end{aligned}\right\}, \tag{17}$$

governing the contaminant concentrations C_1 and C_2 in the layers of fluid with velocities u_1 and u_2. The contaminant mixes from the more concentrated to the less concentrated layer with a mixing time T. Longitudinal diffusion within each layer is described by a diffusivity K. If both T and K are due to eddies that are unresolved in this two-layer description, then they should be related by

$$\ell^2 = 2KT, \tag{18}$$

where ℓ is the thickness of the layers.

The reason for considering this model is to illustrate the relationship of the mean concentration, $\bar{C} = \frac{1}{2}(C_1 + C_2)$, as determined by (17), and the solution of the advection-diffusion equation,

$$\frac{\partial \bar{C}}{\partial t} + \bar{u}\frac{\partial \bar{C}}{\partial x} = K^* \frac{\partial^2 \bar{C}}{\partial x^2}, \tag{19}$$

which should be appropriate when the details of the two layers are averaged out. Here $\bar{u} = \frac{1}{2}(u_1 + u_2)$ is the average velocity and K^* is the enhanced diffusivity.

Equations (17) can be combined to show that $\bar{C}$ must satisfy

$$\left[\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} - K\frac{\partial^2}{\partial x^2}\right)^2 + \frac{2}{T}\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} - K\frac{\partial^2}{\partial x^2}\right) - (\Delta u)^2\frac{\partial^2}{\partial x^2}\right]\bar{C} = 0, \tag{20}$$

where $\Delta u = \frac{1}{2}(u_1 - u_2)$. This is not exactly equation (19); however, if the first term were negligible, then it would be the same as (19) with

$$K^* = K + \frac{1}{2}(\Delta u)^2 T. \tag{21}$$

Careful analysis (Thacker, 1975) can show that ignoring this first term is equivalent to resolving only those changes in C that are slower than the mixing time T and those spatial details that are larger than a mixing length $x = (2K^*T)^{1/2}$. Note that T and x are related according to equation (4), the equation that relates length and time scales for the frost-free turbulence transformation.

On the other hand, for small changes in time, the first term in equation (19) is important and the second term is negligible. The reason for this is that in a short enough time, a negligible amount of mixing between the layers occurs. In this limit, advection within each layer is important and an enhanced diffusivity parameter does not apply. However, each time a bit of contaminant crosses to the other layer, its direction reverses. This gives a long-term net effect of a random walk and diffusion-like behavior. Thus, shear dispersion is like either advection or diffusion, depending upon the scale of the observation. If all of the details of the flow are resolved, shear dispersion is differential advection. But if these details are ignored, which corresponds to filtering out high frequencies and high wavenumbers, then shear dispersion can be represented by an enhanced diffusivity.

Equation (21) can be generalized to the case of turbulent mixing. The shear of the two-layer flow can be thought of as representing an eddy of arbitrary scale in a turbulent flow and the mixing as due to smaller eddies. If the resolution is decreased, then the eddy that represented the shear contributes to the mixing across the shear of a still larger eddy. Thus, the difference $dK \equiv K^* - K$ can be thought of as the increase in eddy diffusivity associated with a decrease in resolution. The factor $(\Delta u)^2$ represents the energy in the scale of the shear, so it should be proportional to $E(k)dk$ or $\Phi(\omega)d\omega$. The mixing time $T = \frac{2\pi}{\omega}$ is related to the diffusivity through equation (4), $\ell^2 = 2KT$, if $\ell = \frac{2\pi}{k}$ is the scale of the shear. Thus, (21) can be generalized to the differential equations

$$dK = -C\frac{E(k)dk}{2Kk^2} = -\frac{C}{2\pi}\frac{\Phi(\omega)d\omega}{\omega}. \tag{22}$$

These equations can be integrated to give equations (5) and (8).

Equations (22) together with (6) are sufficient to determine the frost-free turbulence transformation. To see this, differentiate equation (6) and substitute from (21) to get an equation relating $d\omega$ and dk,

$$d\omega = \pi^{-1}\left[2Kk - C\frac{E(k)}{2K}\right]. \tag{23}$$

Now use (23) to eliminate $d\omega$ and dk from (22). The result is exactly the transformation given by equations (21).

The principal point to be seen from this heuristic derivation is that the eddy diffusivity parameterization should only be valid for an appropriately averaged description of the flow. Such an average should filter out high frequencies and high wavenumbers, where the cut-off values are related by equation (6). The result of this averaging should be expressions like equations (5) and (8) for the diffusivity. It is the extent of the averaging that determines what should be resolved as advection and what should be parameterized as diffusion.

It is interesting to note the similarity of the ideas behind this heuristic derivation of equation (5) and those of Tchen and of Nakano who obtain the same expression for eddy viscosity. Tchen (1973, 1975) uses a hierarchy of ensembles to allow for a varying degree of resolution, which is expressed here as a differential equation. His memory chain corresponds to a hierarchy of time scales that are associated with the hierarchy of length scales. Nakano (1972) bases his theory upon the idea that smaller eddies serve to damp the larger eddies while larger eddies serve to distort and advect smaller eddies. The damping is simply mixing of momentum, so his ideas are the same as those used here.

Recently, McComb (1974) obtained a similar expression for eddy viscosity by modifying Edwards' (1964) theory of turbulence. Their idea is that small scale advection is like random stirring, which is basically the same as the idea presented here. In the two-layer model, the random forces are provided by the shear and the cross-shear mixing. This can be seen clearly from Monin and Yaglom's (1971) derivation of equation (20) as a Fokker-Planck equation for a Markov process.

None of these dynamic theories has yet produced an equation, similar to (8), expressing the eddy viscosity as an integral over the time spectrum. For both (5) and (8) to hold, there must be a correspondence between frequencies and wavenumbers, at least in the statistical sense that an ensemble average should filter out high frequencies and high wavenumbers. Tchen's theory seems to be the closest to this. It would be interesting to see if such an expression can be obtained from these theories. If so, then a more rigorous derivation of the frost-free turbulence transformation should be possible.

6. Summary

The frost-free turbulence transformation presented here in equations (10) is based upon two assumptions. The first is that there is a correspondence between spatial and temporal scales of the turbulence. This is justified by the results of the dye diffusion experiments as summarized by Okubo's (1971) diagrams and by the argument that a mixing length and a relaxation time can be assigned to the averaging process that results in the eddy diffusivity. The second is that an expression such as (5) can be used to describe the scale dependence of the eddy diffusivity. This is justified by the generalization of the results from shear dispersion and by the dynamical theories of Tchen, Nakano, Edwards, and McComb. This transformation successfully relates current records with dye diffusion data and agrees with the results of dimensional arguments. Therefore, this transformation appears to be valid and should be useful as a working hypothesis for further studies of turbulence.

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