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PARABOLIC APPROXIMATIONS FOR GLOBAL ACOUSTIC PROPAGATION MODELING

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CONTENTS

PAGE

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David R. Palmer

ABSTRACT -- *Motivated by the difficulty in using the splitting matrix method to obtain parabolic approximations to complicated wave equations, we have developed an alternative method. It is three dimensional, does not a priori assume a preferred direction or path of propagation in the horizontal, determines spreading factors, and results in equations that are energy conserving. It is an extension of previous work by several authors relating parabolic equations to the horizontal ray acoustics approximation. Unlike previous work it applies the horizontal ray acoustics approximation to the propagator rather than to the Green's function or the homogenous field. The propagator is related to the Green's function by an integral over the famous "fifth parameter" of Fock and Feynman. Methods for evaluating this integral are equivalent to narrow-angle approximations and their wide-angle improvements. When this new method is applied to simple problems it gives the standard results. In this paper it is described by applying it to a problem of current interest—the development of a parabolic approximation for modeling global underwater and atmospheric acoustic propagation. The oceanic or atmospheric waveguide is on an Earth (or other heavenly body) that is modeled as an arbitrary convex solid of revolution. The method results in a parabolic equation that is energy conserving and has a spreading factor that describes field intensification for antipodal propagation. Significantly, it does not have the singularities in its range-sliced version possessed by many parabolic equations developed for global propagation. The work is generalized to allow for refracted geodesics and the possibility the depth dependence of the pressure field can be described by adiabatic normal modes.*

1. INTRODUCTION

In the years following the introduction of the parabolic equation method to sound propagation in the ocean (Tappert and Hardin, 1973; Hardin and Tappert, 1973), a great deal of work was done not only in applying it to propagation problems but also in developing improvements on the method. Approximations and computational schemes were developed that more realistically account for the characteristics of the ocean and its boundaries and reduce the restriction to narrow-angle propagation. Fortunately several excellent early (Tappert, 1977) and recent (Brekhovskikh and Godin, 1999; Lee and Pierce, 1995) reviews are available since the literature is too large to provide citations to all the relevant work.

A common technique for deriving the standard parabolic equation and improvements to it has been the splitting matrix method where a wave equation is factored into two contributions representing forward and backward propagating waves. This method was first developed by H. Bremmer (1951) for the one-dimensional problem and latter generalized to three dimensions by Corones (1975) and Tappert (1977). In this approach it is assumed a particular direction in the ocean medium is singled out by the nature of the source excitation and the dominant portion of the acoustic energy propagates in this direction without significant backscatter. For a discussion of this point-of-view, see Corones, DeFacio and Krueger (1982). The square-root differential operator in the forward-propagating one is then approximated in some fashion to obtain a useful parabolic equation. Many times this approximation is done so as to reduce the dependence of the solution on the reference sound speed thus relaxing the narrow-angle approximation.

Most of these early studies applied the parabolic approximation to the Helmholtz equation

$$\left[\nabla^2 + k_0^2 n^2(\vec{x})\right] p(\vec{x}) = 0 \tag{1}$$

in Cartesian coordinates. Here a source radiates at angular frequency ω, where $k_0 = \omega/c_0$ with c_0 some reference sound speed, $n(\vec{x}) = c_0/c(\vec{x})$ is the index of refraction, and $c(\vec{x})$ is the speed of sound at the point $\vec{x} = (x, y, z)$ in the ocean medium. Sometimes the Laplacian was modified to account for a variable density field, $\nabla^2 \rightarrow \rho(\vec{x})\vec{\nabla} \cdot \left((1/\rho(\vec{x}))\,\vec{\nabla}\right)$, and sometimes currents were included by introducing an effective sound speed equal to the actual sound speed shifted by the component of the current in the direction of propagation.

When one considers more complicated wave equations, however, progress has been slow in developing useful parabolic approximations using the splitting matrix method. We illustrate this with three examples. First is the problem of obtaining parabolic approximations to the elastic wave equation (Corones, DeFacio and Krueger, 1982; Hudson, 1980; Landers and Claerbout, 1972; Wales and McCoy, 1983; Greene, 1985; Wetton and Brooke, 1990; Collins, 1991; Collins, 1993a). Although considerable effort has been devoted to this problem, parabolic equations have only been developed under limiting conditions that include consideration of a two-dimensional rather than a three-dimensional medium, Lamé parameters that are range independent or have very small spatial gradients, or very special types of propagation.

Second is the problem of obtaining parabolic equations for propagation in a moving inhomogeneous fluid. As a result of the work by Brekhovskikh and Godin (1999) and Godin (1987), it is now known that many of the earlier approximations, including the use of an effective sound speed, cannot adequately account for the effects of the motion in many important and practical situations. The appropriate wave equation, correct to first order in the Mach number, is complicated and considerable ingenuity has been devoted developing parabolic approximations that are wide-angle, conserve energy, obey the flow reversal theorem and boundary conditions consistent with the parabolic approximation (a requirement ignored in most studies) (Godin, 1991; Godin and Mokhov, 1992; Godin, 1998a; 1998b; 1999). Most progress on this problem was obtained using the multiple-scales approach (see below) rather than with the splitting matrix method because of the problems associated with the splitting matrix method discussed here (A. O. Godin, private conversation, 1998).

A final example is the problem of developing parabolic approximations for wave equations in curvilinear coordinates with the purpose of modeling global acoustic propagation (Collins, 1993b; McDonald *et al.*, 1994; Collins *et al.*, 1995; 1996). It is common to reduce the problem to a two-dimensional one by introducing local normal modes and by ignoring, for the most part, mode coupling. The matrix splitting method is then applied to the resulting equation in the horizontal coordinates to obtain a forward-propagating parabolic equation. The range-sliced version of this equation, needed to obtain a marching algorithm, has singularities (Collins *et al.*, 1996). These singularities result from the fact that the commutators in the Baker-Campbell-Hausdorff formula used to develop the range-sliced expression cannot be dropped for small range intervals. They actually become more singular with order, regardless of the size of the range interval. The range-sliced marching algorithm, as well as the closely-related range-sliced path integral representation, are only valid in Cartesian coordinates. This point has been discussed in detail in the literature. See, for example, Böhm and Junker, (1987), where many important papers on this subject are cited including those describing techniques for scaling variables to eliminate the singularities. These scaling techniques are only now being applied to wave propagation problems. This problem of singularities would exist even if the three-dimensional wave equation had been considered rather than the reduced two-dimensional one.

For these complicated wave equations, the splitting matrix method is not very easy to apply. It is not always obvious how the factorization should be done nor is it obvious how the resulting differential operators should be approximated to obtain wide-angle equations. Normalizations and spreading factors are not easy to obtain. (The quantity $1/\sqrt{r}$ in Eq.(2) below is a spreading factor.) With few exceptions, e.g., Tappert, Spiesberger, and Boden (1995), studies that use the splitting matrix method ignore the determination of the spreading factor.

It seems worthwhile then to consider new approaches to developing parabolic approximations that might have application to complicated wave equations. In developing them we are guided by four principles (prejudices actually):

A. One should start with equations that are three dimensional. By starting with two-dimensional equations one has already made an assumption about the nature of the propagation. One has assumed there is no scattering of the acoustic field out of the surface defined by the two coordinates. If this is a valid assumption, it should be a consequence of the process of making the parabolic approximation and not something imposed on the problem from the outset and considered as a separate, unjustified approximation. One should also not approach the three-dimensional problem by patching together solutions to the two-dimensional one. One should be able to start with the three-dimensional problem and show that its solution can be written, with appropriate approximations, in terms of solutions to the two-dimensional problem if this patching process is valid.

B. One should not assume any preferred horizontal direction or path for the propagation. In Corones, DeFacio and Krueger, (1982), one reads, "*...the parabolic approximation is used when a particular direction is singled out in the medium by the nature of the excitation...The direction is distinguished by the excitation, not by the medium.*" While this statement may be valid for laser propagation through the atmosphere where the radiation is emitted in a narrow beam, it cannot be valid for low-frequency sound propagation in the ocean because low-frequency sources (excitations) are essentially omnidirectional. If there is a preferred direction or path it should be defined by the characteristics of the medium and should be determined in the process of making the parabolic approximation.

When one writes a wave equation in terms of cylindrical coordinates (r,φ,z) and discards derivatives with respect to φ, one has already assumed a preferred direction. It is the direction of the straight line along the radial connecting the source to the receiver. When one writes

$$p(r,\varphi,z) = \frac{e^{ik_0 r}}{\sqrt{r}}\psi(r,\varphi,z) \tag{2}$$

for the pressure field and assumes ψ varies slowly with r, one has also defined a preferred direction. This is an important point because more complicated wave equations may not have a preferred direction along the radial but in some other direction in the horizontal plane. If horizontal multipaths are present the preferred direction is not unique or it is undefined.

C. The approach should provide a way for determining spreading factors. While for some applications spreading factors are not important, it is difficult to imagine a systematic, general method for developing parabolic equations that only determines part of the field.

D. A general parabolic approximation method should consist of two distinct types of approximations: those related to horizontal variation; a preferred direction of propagation along a horizontal multipath, no backscattering, no out-of-plane propagation, etc. and those related to vertical variation; narrow-angle and wide-angle approximations. When one considers the application of the parabolic method to simple problems these two types of approximations are distinct. For example, only the first type is needed to develop a parabolic equation for the propagation of a single adiabatic mode in a range-dependent waveguide while only the second type is needed for propagation of a field in a range-independent waveguide described by a modal sum. One might expect that a method applicable to more complicated problems as well as these simple ones would not mix the two types of approximations.

In developing an alternative to the splitting matrix method based on the above considerations we are motivated by three observations. First, by working order-by-order in perturbation theory in the range-dependent part of the index of refraction, it was found many years ago (Palmer, 1976) that the solution to the Helmhotz equation could be written in terms of the solution to the standard parabolic equation by making a horizontal eikonal approximation followed by the stationary-phase approximation of an integral over the "fifth parameter" of Fock, (1937) and Feynman, (1951). The stationary-phase approximation was shown to be equivalent to the narrow-angle approximation. A completely different approach, based on the use of path integrals rather than perturbation theory, gave the same result (Palmer, 1979). (In this case, horizontal ray acoustics was assumed rather than the closely related horizontal eikonal approximation.) Second, an alternative to the splitting matrix method is the method of multiple scales in which the horizontal variables are scaled differently from the depth variable (Tappert, 1977; Siegmann, Kreigsmann, and Lee, 1985; Kreigsmann, 1985; Orchard, Siegmann, and Jacobson, 1992). This method leads to the factorization of the field into a function that obeys the parabolic equation (e.g., ψ) and a kinematic factor that is dependent on only the horizontal coordinates (e.g., $\exp(ik_0 r)/\sqrt{r}$). The multiple scales method is closely related to the method used by Brekhovskikh and Godin (1999, Section 7.2) to develop horizontal ray theory for a weakly range-dependent, three-dimensional medium. Finally, it is well known that there is a close relationship between ray acoustics (geometric optics) and parabolic equations (Myers and McAninch, 1978; McAninch, 1986; Babič and Buldyrev, 1991, Chap. 6).

The method we propose then amounts to a horizontal ray acoustics approximation followed by the approximation of the integral over the Fock-Feynman parameter. It satisfies the four principles we discussed above. Its validity is based on the fact that horizontal scales of variability in the ocean are much greater than vertical ones.

The only aspect of this method that is different from what others have done is that we apply the horizontal ray acoustics approximation not to the wave equation but to the equation satisfied by the propagator. The two are related by an integral over the Fock-Feynman parameter. This is the key aspect of the method because this propagator obeys (exactly) a four-coordinate parabolic equation. Approximations such as the horizontal ray acoustics approximation essentially reduce its dimensionality.

To illustrate this method we apply it to the problem of developing a parabolic approximation for the wave equation in curvilinear coordinates appropriate for modeling global acoustic propagation. We do not assume the Earth is spherical or even ellipsoidal but only that it is a convex solid of revolution. In this illustative example, we assume that currents can be ignored.

This report is organized as follows. In the next section we characterize the ocean wave guide on an Earth modeled as a solid of revolution. In Section 3 the propagator for the Helmholtz equation (modified to include a deep-dependent density variation) and the Fock-Feynman parameter are introduced. In Section 4 we factor the propagator and apply the horizontal ray acoustics approximation. The eikonal and transport equations are derived and solved. We provide a detailed solution to the transport equation that determines the spreading factor for this problem. Section 5 treats the vertical equation and the stationary phase approximation. When applied to the simple Helmhotz equation in Cartesian coordinates it produces the standard narrow-angle parabolic equation. In many situations one wants to do better. One extension would be to include the possibility of horizontal refraction and horizontal mutipaths. Another extension would be the description of the depth dependence of the pressure field by use of normal modes. Section 6 contains of discussion of these possibilities. Finally, in Section 7 we summarize the approach used in the paper.

2. THE EARTH AS A SOLID OF REVOLUTION

The Earth is assumed to be a solid of revolution with, when viewed from space, a convex surface everywhere. The origin of the Cartesian coordinate system is centered in the Earth with the axis of rotation the z-axis. The position vector from the origin of this coordinate system to a point on the Earth's surface is

$$\vec{x}_S = x_S\ \hat{x} + y_S\ \hat{y} + z_S\ \hat{z} \tag{3}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors along the coordinate axes. Since the axis of rotation is the z-axis, one can write

$$x_S = \rho_S(\varphi_g)\cos\lambda$$
$$y_S = \rho_S(\varphi_g)\sin\lambda$$

where φ_g is the *geocentric latitude*

$$\varphi_g \equiv \arctan\left(\frac{z_S}{\sqrt{x_S^2 + y_S^2}}\right)$$

and λ is the longitude.

2.1 Unit vectors

Consider the directed line segment from the point on the surface with coordinates (φ_g, λ) to the nearby point $(\varphi_g + \delta\varphi_g, \lambda + \delta\lambda)$,

$$\delta\vec{x}_S = \delta x_S\ \hat{x} + \delta y_S\ \hat{y} + \delta z_S\ \hat{z} \tag{4}$$

where $\delta x_S = x_S(\varphi_g + \delta\varphi_g, \lambda + \delta\lambda) - x_S(\varphi_g, \lambda)$, etc. To second order in the quantities $\delta\varphi_g$ and $\delta\lambda$ we have

$$\delta x_S = \rho_S' \cos\lambda\, \delta\varphi_g - \rho_S \sin\lambda\, \delta\lambda$$

$$+\frac{1}{2}\rho_S'' \cos\lambda\ \left(\delta\varphi_g\right)^2 - \frac{1}{2}\rho_S \cos\lambda\ (\delta\lambda)^2 - \rho_S' \sin\lambda\, \delta\varphi_g\ \delta\lambda$$

$$\delta y_S = \rho_S' \sin\lambda\, \delta\varphi_g + \rho_S \cos\lambda\, \delta\lambda$$

$$+\frac{1}{2}\rho_S'' \sin\lambda\ \left(\delta\varphi_g\right)^2 - \frac{1}{2}\rho_S \sin\lambda\ (\delta\lambda)^2 + \rho_S' \cos\lambda\, \delta\varphi_g\ \delta\lambda$$

$$\delta z_S = z_S'\, \delta\varphi_g + \frac{1}{2} z_S''\ \left(\delta\varphi_g\right)^2$$

where a prime indicates differentiation with respect to φ_g. The terms linear in the infinitesimals are used to derive expressions for unit vectors and the quadratic terms are used to derive expressions for radii of curvature.

The unit vector tangent to the surface and in the direction of increasing λ is

$$\hat{\lambda} = \frac{\partial\vec{x}_S}{\partial\lambda} \Big/ \left|\frac{\partial\vec{x}_S}{\partial\lambda}\right| = -\sin\lambda\ \hat{x} + \cos\lambda\ \hat{y} \tag{5}$$

It is convenient to define a second unit vector in the x-y plane orthogonal to $\hat{\lambda}$,

$$\hat{\rho}_S = \cos\lambda\ \hat{x} + \sin\lambda\ \hat{y} \tag{6}$$

The unit vector tangent to the surface in the direction of increasing φ_g is

$$\hat{\varphi}_g = \frac{\partial\vec{x}_S}{\partial\varphi_g} \Big/ \left|\frac{\partial\vec{x}_S}{\partial\varphi_g}\right| = \frac{\rho_S'\ \hat{\rho}_S + z_S'\ \hat{z}}{\xi} \tag{7}$$

with

$$\xi \equiv \sqrt{(\rho'_S)^2 + (z'_S)^2} \tag{8}$$

Finally, the unit vector normal to the surface is

$$\hat{n} = \hat{\lambda} \times \hat{\varphi}_g = \frac{z'_S\ \hat{\rho}_S - \rho'_S\ \hat{z}}{\xi} \tag{9}$$

In terms of these unit vectors we have

$$\vec{x}_S = \rho_S(\varphi_g)\ \hat{\rho}_S + z_S(\varphi_g)\ \hat{z} \tag{10}$$

$$\hat{x}_S = \frac{\vec{x}_S}{|\vec{x}_S|} = \cos\varphi_g\ \hat{\rho}_S + \sin\varphi_g\ \hat{z} \tag{11}$$

and

$$\begin{aligned} \delta\vec{x}_S &= \xi\delta\varphi_g\ \hat{\varphi}_g + (\rho_S + \rho'_S\delta\varphi_g)\delta\lambda\ \hat{\lambda} \\ &+ \frac{1}{2\xi}\left[(\rho'_S\rho''_S + z'_S z''_S)\left(\delta\varphi_g\right)^2 - \rho_S\rho'_S\left(\delta\lambda\right)^2\right]\ \hat{\varphi}_g \\ &+ \frac{1}{2\xi}\left[(z'_S\rho''_S - \rho'_S z''_S)\left(\delta\varphi_g\right)^2 - \rho_S z'_S\left(\delta\lambda\right)^2\right]\ \hat{n} \end{aligned} \tag{12}$$

The differential line segment tangent to the surface is

$$d\vec{x}_S = \xi\ d\varphi_g\ \hat{\varphi}_g + \rho_S\ d\lambda\ \hat{\lambda} \tag{13}$$

and the differential arc length is

$$ds \equiv |d\vec{x}_S| = \sqrt{\xi^2(d\varphi_g)^2 + \rho_S^2(d\lambda)^2} \tag{14}$$

2.2 Radii of Curvature

At a point on the surface (φ_g, λ), there are two principal radii of curvature; the *meridional*, describing curvature in the $\hat{\varphi}_g - \hat{n}$ plane and the *prime vertical*, describing curvature in the $\hat{\lambda} - \hat{n}$ plane. We discuss first the meridional. Consider two points in the meridional plane at (φ_g, λ) and at $(\varphi_g + \delta\varphi_g, \lambda)$. The directed line segment from the first point to the second can be written as

$$\delta\vec{x}_S = \delta x_\varphi\ \hat{\varphi}_g + \delta x_n\ \hat{n} \tag{15}$$

The meridional radius of curvature μ is given by the expression

$$\mu = -\frac{1}{2} Lim_{\delta\varphi_g \to 0} \frac{\left(\delta x_\varphi\right)^2}{\delta x_n} \tag{16}$$

provided the limit exists. If $\mu > 0$ the surface is convex at the point (φ_g, λ) with respect to the origin (like a sphere's outer surface) and if $\mu < 0$ the surface is concave. By setting $\delta\lambda = 0$ in Eq.(12) we find

$$\delta x_\varphi = \xi \delta\varphi_g + \frac{1}{2\xi}\left(\rho'_S \rho''_S + z'_S z''_S\right)\left(\delta\varphi_g\right)^2$$

and

$$\delta x_n = \frac{1}{2\xi}\left(z'_S \rho''_S - \rho'_S z''_S\right)\left(\delta\varphi_g\right)^2$$

Substituting into Eq.(16) gives

$$\mu = \frac{\xi^3}{\rho'_S z''_S - z'_S \rho''_S} \tag{17}$$

The prime vertical radius of curvature is determined in a similar fashion to be

$$\nu = \frac{\rho_S\, \xi}{z'_S} \tag{18}$$

The differential line segment, Eq.(13), can now be written

$$d\vec{x}_S = \frac{\mu}{\xi^2}\left(\rho'_S z''_S - z'_S \rho''_S\right) d\varphi_g\ \hat{\varphi}_g + \frac{\nu z'_S}{\xi} d\lambda\ \hat{\lambda} \tag{19}$$

2.3 Geodetic Latitude

The geocentric latitude is the angle between the position vector and the equatorial plane

$$\varphi_g = \arcsin\left(\hat{x}_S \cdot \hat{z}\right) \tag{20}$$

The *geodetic latitude* φ is defined to be the angle the normal to the surface makes with respect to the equatorial plane

$$\varphi = \arcsin\left(\hat{n} \cdot \hat{z}\right) \tag{21}$$

Clearly

$$\sin\varphi = -\frac{\rho'_S}{\xi} \quad ; \quad \cos\varphi = \frac{z'_S}{\xi} \tag{22}$$

The geocentric and geodetic latitudes are equal only if the Earth is modeled as a sphere. To see this we note that if $\varphi_g = \varphi$, then

$$\frac{z_S}{\rho_S} = \tan\varphi_g = \tan\varphi = \frac{-\rho'_S}{z'_S}$$

or

$$\frac{d}{d\varphi_g}\left(\rho_S^2+z_S^2\right)=0$$

giving the defining equation for a sphere

$$x_S^2+y_S^2+z_S^2=\text{constant}$$

In terms of the geodetic latitude

$$\hat{\varphi}_g=-\sin\varphi\ \hat{\rho}_S+\cos\varphi\ \hat{z}$$

and

$$\hat{n}=\cos\varphi\ \hat{\rho}_S+\sin\varphi\ \hat{z}$$

From Eq.(13),

$$\frac{d\vec{x}_S}{d\varphi}=\frac{\xi}{\varphi'}\hat{\varphi}_g$$

so

$$\hat{\varphi}=\pm\hat{\varphi}_g \tag{23}$$

where the sign is determined by the sign of $\varphi'=d\varphi/d\varphi_g$ since $\xi>0$. This derivative can be expressed in terms of the derivatives of ρ_S and z_S by differentiating both sides of the equation $\tan\varphi=-\rho'_S/z'_S$ with respect to φ_g. That is

$$\varphi'=-\frac{\sin\varphi\ z''_S+\cos\varphi\ \rho''_S}{\xi}=\frac{\rho'_S z''_S-\rho''_S z'_S}{\xi^2}$$

or

$$\mu\varphi'=\xi \tag{24}$$

Since we are assuming the Earth has a convex surface everywhere $\mu>0$. Therefore $\varphi'>0$, $\hat{\varphi}=\hat{\varphi}_g$, and there is a one-to-one relationship between the two latitudes enabling us to replace φ_g with φ as the independent variable. This is an advantage since oceanographic data are referenced to the geodetic rather than the geocentric latitude.

The basic coordinate transformation equations can be re-written as

$$x_S=\nu\cos\varphi\cos\lambda \tag{25}$$

$$y_S=\nu\cos\varphi\sin\lambda \tag{26}$$

$$z_S=\chi\sin\varphi \tag{27}$$

The differential line segment is

$$d\vec{x}_S=\mu\ d\varphi\ \hat{\varphi}+\nu\cos\varphi\ d\lambda\ \hat{\lambda} \tag{28}$$

Observing from this relation that

$$\mu^2 = \left| \frac{d\vec{x}_S}{d\varphi} \right|^2$$

we obtain

$$\mu^2 = \left(\frac{d}{d\varphi} (\nu \cos\varphi) \right)^2 + \left(\frac{d}{d\varphi} (\chi \sin\varphi) \right)^2 \tag{29}$$

Since

$$z'_S \sin\varphi + \rho'_S \cos\varphi = 0$$

we have

$$\sin\varphi \frac{d\,(\chi \sin\varphi)}{d\varphi} + \cos\varphi \frac{d\,(\nu \cos\varphi)}{d\varphi} = 0$$

These equations yield

$$\mu = \nu \left| 1 - \frac{\cot\varphi}{\nu} \frac{d\nu}{d\varphi} \right| \tag{30}$$

and the useful relations

$$\frac{d}{d\varphi} (\nu \cos\varphi) = -\mu \sin\varphi \quad ; \quad \frac{d}{d\varphi} (\chi \sin\varphi) = +\mu \cos\varphi \tag{31}$$

The arc length is

$$ds = \sqrt{\mu^2 \,(d\varphi)^2 + \nu^2 \cos^2\varphi \,\ (d\lambda)^2} \tag{32}$$

The unit vector directed from the point with coordinates (φ, λ) to the one with coordinates $(\varphi + d\varphi, \lambda + d\lambda)$ is

$$\hat{s} = \frac{d\vec{x}_S}{ds} = \mu \, \frac{d\varphi}{ds} \, \hat{\varphi} + \nu \cos\varphi \, \frac{d\lambda}{ds} \, \hat{\lambda} \tag{33}$$

The unit vector $\hat{s}$ can also be written in terms of the angle α the differential line segment makes with respect to $\hat{\varphi}$, i.e., the angle between $d\vec{x}_S$ and "north"

$$\hat{s} = \cos\alpha \ \hat{\varphi} + \sin\alpha \ \hat{\lambda} \tag{34}$$

where $\cos\,\alpha = \mu \ (d\varphi/ds)$ and $\sin\alpha = \nu \cos\varphi \ (d\lambda/ds)$.

We define

$$R(\varphi) \equiv \sqrt{\mu(\varphi)\nu(\varphi)} \tag{35}$$

The length $R(\varphi)$ will play an important role in the development of the parabolic approximation. Since the Earth is almost spherical, it is possible to write

$$\frac{1}{R(\varphi)} = \frac{1}{\bar{R}} \left(1 + \eta g(\varphi)\right) \tag{36}$$

where $\bar{R}$ is the mean radius of the Earth and g is a function of order unity. The small parameter η is a measure of the deviation of the solid from a sphere. If the Earth is taken to be an ellipsoid, η would be the eccentricity squared, $\eta \approx 1/150$ (see, e.g., J. Dworski and J. A. Mercer, 1990).

2.4 Unit vectors revisited

We list here the unit vectors that have been introduced with their differentials. We have

$$\hat{\varphi} = -\sin\varphi\ \hat{\rho}_S + \cos\varphi\ \hat{z} \tag{37}$$

$$\hat{\lambda} = -\sin\lambda\ \hat{x} + \cos\lambda\ \hat{y} \tag{38}$$

$$\hat{n} = \cos\varphi\ \hat{\rho}_S + \sin\varphi\ \hat{z} \tag{39}$$

$$\begin{aligned} \hat{\rho}_S &= \cos\lambda\ \hat{x} + \sin\lambda\ \hat{y} \\ &= \cos\varphi\ \hat{n} - \sin\varphi\ \hat{\varphi} \end{aligned} \tag{40}$$

For the differentials we have

$$d\hat{\varphi} = -\sin\varphi\ \hat{\lambda}\ d\lambda - \hat{n}\ d\varphi \tag{41}$$

$$d\hat{\lambda} = -\hat{\rho}_S\ d\lambda \tag{42}$$

$$d\hat{n} = \cos\varphi\ \hat{\lambda}\ d\lambda + \hat{\varphi}\ d\varphi \tag{43}$$

and

$$d\hat{\rho}_S = \hat{\lambda}\ d\lambda \tag{44}$$

2.5 The Laplacian operator and the delta function

The position vector to a general point in the ocean, below the surface, is

$$\vec{x} = \vec{x}_S - \zeta\hat{n}$$

where $\vec{x}_S$ is the position vector of the point on the surface directly over the point of interest, ζ (≥ 0) is the depth at that point. Propagation in the atmosphere can be considered by simply changing the sign of ζ. The differential line element is

$$d\vec{x} = d\vec{x}_S - d\zeta\ \hat{n} - \zeta\ d\hat{n}$$

$$= (\mu - \zeta)\ d\varphi\ \hat{\varphi} + (\nu - \zeta)\cos\varphi\ d\lambda\ \hat{\lambda} - d\zeta\ \hat{n}$$

where we have used Eq.(43). The differential volume element is

$$d^3\vec{x} = (\mu - \zeta)(\nu - \zeta)\cos\varphi\ d\varphi d\lambda d\zeta$$

Also

$$\nabla^2 = \frac{1}{(\mu-\zeta)(\nu-\zeta)\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\nu-\zeta}{\mu-\zeta}\cos\varphi\,\frac{\partial}{\partial\varphi}\right)$$
$$+\frac{1}{(\mu-\zeta)^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{(\mu-\zeta)(\nu-\zeta)}\frac{\partial}{\partial\zeta}\left((\mu-\zeta)(\nu-\zeta)\frac{\partial}{\partial\zeta}\right) \quad (45)$$

and

$$\delta^{(3)}\left(\vec{x}-\vec{x}_0\right) = \frac{1}{(\mu-\zeta)(\nu-\zeta)}\delta\left(\sin\varphi-\sin\varphi_0\right)\delta\left(\lambda-\lambda_0\right)\delta\left(\zeta-\zeta_0\right)$$

In writing these relations we have used the fact that ν and μ are functions of φ but not of λ and ζ. For many problems it is not ∇^2 which is of concern but rather $\rho\nabla\cdot(\rho^{-1}\nabla)$ where ρ is the density of the medium. Because of stratification ρ is usually assumed to depend only on the depth variable ζ. If this is the case, the last term in Eq.(45) is replaced with

$$\frac{\rho(\zeta)}{(\mu-\zeta)(\nu-\zeta)}\frac{\partial}{\partial\zeta}\left(\frac{(\mu-\zeta)(\nu-\zeta)}{\rho(\zeta)}\frac{\partial}{\partial\zeta}\right)$$

In most work on global propagation, a "thin-ocean" approximation is made. That is, since the depth of the ocean is much less than the radii of curvature, $\zeta \ll \nu$ and $\zeta \ll \mu$, it is valid to replace $\nu-\zeta$ with ν and $\mu-\zeta$ with μ everywhere in the expressions above, even before differentiation with respect to ζ or φ. With this approximation, we have

$$d\vec{x} = \mu d\varphi\,\hat{\varphi} + \nu\cos\varphi\,d\lambda\,\hat{\lambda} - d\zeta\,\hat{n} \quad (46)$$

and

$$d^3\vec{x} = J(\varphi)d\varphi d\lambda d\zeta \quad (47)$$

where

$$J(\varphi) \equiv \mu\nu\cos\varphi \quad (48)$$

Moreover

$$\rho\nabla\cdot\left(\rho^{-1}\nabla\right) = \nabla_S^2 + \rho\frac{\partial}{\partial\zeta}\left(\frac{1}{\rho}\frac{\partial}{\partial\zeta}\right) \quad (49)$$

where

$$\nabla_S^2 = \frac{1}{J(\varphi)}\left[\frac{\partial}{\partial\varphi}\left(\frac{\nu\cos\varphi}{\mu}\frac{\partial}{\partial\varphi}\right)+\frac{\mu}{\nu\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\right] \quad (50)$$

and

$$\delta^{(3)}\left(\vec{x}-\vec{x}_0\right) = \frac{1}{J(\varphi)}\delta\left(\varphi-\varphi_0\right)\delta\left(\lambda-\lambda_0\right)\delta\left(\zeta-\zeta_0\right) \quad (51)$$

The "thin-ocean" approximation can be relaxed by scaling and redefining the index of refraction, See e.g., Tappert (1977) or Collins (1993b). If we had done this scaling the analysis would be unchanged, we would have simplied ended up with expressions involving the redefined index of refraction.

3. GREEN'S FUNCTION AND THE PROPAGATOR

The equation for G the Green's function or impulse response function for the time-independent problem is

$$\left[\rho(\vec{x})\vec{\nabla}\cdot\left(\frac{1}{\rho(\vec{x})}\vec{\nabla}\right)+k_0^2 n^2(\vec{x})\right]G(\vec{x}\,|\,\vec{x}_0)=-\delta^{(3)}\left(\vec{x}-\vec{x}_0\right) \tag{52}$$

for a point source located at $\vec{x}_0=(x_0,y_0,z_0)$. The Green's function will also satisfy appropriate boundary conditions at the surface and bottom of the ocean. In developing parabolic approximations, many authors begin with the wave equation appropriate for a source-free region of space (e.g., Eq.(1)) . We prefer to begin with Eq.(52) so that a separate analysis is not needed to account for a source. This analysis involves solving for the field on a surface close to the source and then matching boundary conditions on that surface. It should be noted in this regard that general, linear equations for sound propagation were derived in Brekhovskikh and Godin, (1999), (Eqs. 4.1.9-4.1.11) by including from the outset acoustic source terms.

It is not Green's functions, however, that obey parabolic equations and marching algorithms, but propagators. The two are related by an integral over τ, the Fock-Feynman parameter (Fock, 1937; Feynman, 1951)

$$G(\vec{x}\,|\,\vec{x}_0)=\frac{i}{2k_0}\int_0^\infty d\tau\, e^{i\tau k_0/2}\Phi(\tau;\vec{x}\,) \tag{53}$$

The propagator function Φ is defined by the equations

$$\left[2ik_0\frac{\partial}{\partial\tau}+\rho\vec{\nabla}\cdot\left(\frac{1}{\rho}\vec{\nabla}\right)-2k_0^2V(\vec{x})\right]\Phi(\tau;\vec{x}\,)=0\ ;\ \tau>0 \tag{54}$$

and

$$Lim_{\tau\to 0}\Phi(\tau;\vec{x}\,)=\delta^{(3)}\left(\vec{x}-\vec{x}_0\right) \tag{55}$$

where the sound speed variation is

$$V(\vec{x})\equiv-\frac{1}{2}\left[n^2(\vec{x})-1\right]$$

Equation (53) is easy to verify using integration by parts. Convergence is assured by assuming, as always, that the wave number k_0 has an infinitesimal, positive, imaginary part. The functions Φ and G obey the same boundary conditions at the surface and bottom of the ocean.

In our view, all parabolic approximation methods should start with equations analogous to Eqs.(53)-(55). Approximations to Φ should then be made based on the physics of the problem. For example, for laser propagation in the atmosphere, there is a preferred direction determined by the source beam whereas for low-frequency sound propagation in the ocean, horizontal spatial scales of variability are some two-orders of magnitude greater than vertical scales and always much larger than the wavelength. Such physical considerations justify approximations to Φ.

In terms of the curvilinear coordinates introducted in the previous section, Eqs. (54) and (55) become

$$\left[2ik_0\frac{\partial}{\partial\tau}+\nabla_S^2+\rho\frac{\partial}{\partial\zeta}\left(\frac{1}{\rho}\frac{\partial}{\partial\zeta}\right)-2k_0^2V(\varphi,\lambda,\zeta)\right]\Phi(\tau;\varphi,\lambda,\zeta)=0 \qquad (56)$$

and

$$Lim_{\tau\to 0}\Phi(\tau;\varphi,\lambda,\zeta)=\frac{1}{J(\varphi)}\delta\left(\varphi-\varphi_0\right)\delta\left(\lambda-\lambda_0\right)\delta\left(\zeta-\zeta_0\right) \qquad (57)$$

with ∇_S^2 given by Eq.(50).

4. HORIZONTAL RAY ACOUSTICS

4.1 Separability and the ray acoustics approximation

We write Φ as the product of two factors

$$\Phi = H\Psi \qquad (58)$$

and require that H be independent of the depth coordinate and obey a parabolic equation in the horizontal coordinates

$$\left[2ik_0\frac{\partial}{\partial\tau}+\nabla_S^2\right]H(\tau,\varphi,\lambda)=0 \qquad (59)$$

$$Lim_{\tau\to 0}H(\tau,\varphi,\lambda)=\frac{1}{J(\varphi_0)}\delta\left(\varphi-\varphi_0\right)\delta\left(\lambda-\lambda_0\right) \qquad (60)$$

As we shall see, the field H will determine spreading factors and the path along which the field Ψ propagates. The field Ψ , on the other hand, will reflect the dynamic effects associated with a variable sound speed. The factorization in Eq.(58) of the solution to a wave equation into functions having different coordinate dependences has been used by many authors in the development of parabolic approximations (Myers and McAninch, 1978; McAninch, 1986; Babič and Buldyrev, 1991). It is analogous to the factorization of the solution to the Helmholtz equation (Eq.(2)) that is often the starting point in the development of the standard parabolic approximation for the two dimensional propagation problem in range and depth. In the path integral approach (Palmer, 1978; 1979) where it is as easy to develop ray acoustics approximations in one or two of the three coordinates as in all three, the factorization is not an *Ansätz* but rather a consequence of making the horizontal ray acoustics approximation.

We assume horizontal spatial scales are large enough compared to the acoustic wavelength that the ray acoustics approximation is valid when considering horizontal wave motion. We do not make any assumption about the vertical scales of variability. Following the general procedure developed by Weinberg (1962) we write Eq.(59) as a system of first order equations

$$\begin{bmatrix} 2ik_0 J\partial/\partial\tau & \partial/\partial\varphi & \partial/\partial\lambda \\ \partial/\partial\varphi & -\mu/\left(\nu\cos\varphi\right) & 0 \\ \partial/\partial\lambda & 0 & -\left(\nu\cos\varphi\right)/\mu \end{bmatrix} \begin{bmatrix} H \\ H_\varphi \\ H_\lambda \end{bmatrix} = 0 \tag{61}$$

and assume a common exponential factor

$$\begin{bmatrix} H \\ H_\varphi \\ H_\lambda \end{bmatrix} = \begin{bmatrix} \hat{H} \\ \hat{H}_\varphi \\ \hat{H}_\lambda \end{bmatrix} \exp(ik_0 A) \tag{62}$$

so that

$$\begin{bmatrix} 2ik_0 J\left(ik_0\partial A/\partial\tau + \partial/\partial\tau\right) & ik_0\partial A/\partial\varphi + \partial/\partial\varphi & ik_0\partial A/\partial\lambda + \partial/\partial\lambda \\ ik_0\partial A/\partial\varphi + \partial/\partial\varphi & -\mu/\left(\nu\cos\varphi\right) & 0 \\ ik_0\partial A/\partial\lambda + \partial/\partial\lambda & 0 & -\nu\cos\varphi/\mu \end{bmatrix}$$

$$\times \begin{bmatrix} \hat{H} \\ \hat{H}_\varphi \\ \hat{H}_\lambda \end{bmatrix} = 0 \tag{63}$$

Consider one element in the square array above, say the element in the top row and middle column, $ik_0\partial A/\partial\varphi + \partial/\partial\varphi$. This element will contribute to the differential equation the expression

$$\hat{H}\left(ik_0\frac{\partial A}{\partial\varphi} + \frac{\partial\ln\hat{H}}{\partial\varphi}\right)$$

In this matrix approach, the rule for developing the lowest-order ray acoustics approximation is to ignore in the matrix elements gradients of $\hat{H}$ (or $\hat{H}_\varphi$ and $\hat{H}_\lambda$) in comparison to terms involving k_0 times the corresponding gradients of A. This rule is justified by using the traditional approach, i.e., writing the matrix equation as a single equation, expanding the $\hat{H}$ in a series in $1/k_0$, separating real and imaginary parts and equating like powers of k_0. It cannot be justified by considering individual matrix elements such as the one above because the terms involving A and $\hat{H}$ are in general complex. In the present case the advantage of using the matrix approach over the tranditional approach is that it is very easy to apply the ray acoustics approximation to individual coordinates separately. This will become clear when we consider the ray acoustics approximation for Ψ.

Applying this rule to Eq.(63), we obtain, in lowest order, the two equations

$$\begin{bmatrix} -2k_0^2 J\partial A/\partial\tau & ik_0\partial A/\partial\varphi & ik_0\partial A/\partial\lambda \\ ik_0\partial A/\partial\varphi & -\mu/(\nu\cos\varphi) & 0 \\ ik_0\partial A/\partial\lambda & 0 & -\nu\cos\varphi/\mu \end{bmatrix} \begin{bmatrix} \hat{H} \\ \hat{H}_\varphi \\ \hat{H}_\lambda \end{bmatrix} = 0 \tag{64}$$

and

$$[\,\hat{H} \quad \hat{H}_\varphi \quad \hat{H}_\lambda\,] \begin{bmatrix} 2ik_0 J\partial/\partial\tau & \partial/\partial\varphi & \partial/\partial\lambda \\ \partial/\partial\varphi & 0 & 0 \\ \partial/\partial\lambda & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{H} \\ \hat{H}_\varphi \\ \hat{H}_\lambda \end{bmatrix} = 0 \tag{65}$$

Equation (64) gives the *eikonal equation*

$$\frac{\partial A}{\partial\tau} + \frac{1}{2\mu^2}\left(\frac{\partial A}{\partial\varphi}\right)^2 + \frac{1}{2\nu^2\cos^2\varphi}\left(\frac{\partial A}{\partial\lambda}\right)^2 = 0 \tag{66}$$

which determines A. The factor $\hat{H}$ is determined by the *transport equation*

$$\left(\frac{\partial}{\partial\tau} + \frac{1}{\mu^2}\frac{\partial A}{\partial\varphi}\frac{\partial}{\partial\varphi} + \frac{1}{\nu^2\cos^2\varphi}\frac{\partial A}{\partial\lambda}\frac{\partial}{\partial\lambda}\right)\ln\hat{H}^2 + \nabla_S^2 A = 0 \tag{67}$$

which follows from Eq.(65) after Eq.(64) is used to express $\hat{H}_\varphi$ and $\hat{H}_\lambda$ in terms of $\hat{H}$,

$$\hat{H}_\varphi = ik_0\frac{\nu\cos\varphi}{\mu}\frac{\partial A}{\partial\varphi}\hat{H}\ ;\ \hat{H}_\lambda = ik_0\frac{\mu}{\nu\cos\varphi}\frac{\partial A}{\partial\lambda}\hat{H} \tag{68}$$

4.2 Solving the eikonal equation: ray paths

The eikonal equation, Eq.(66), is solved in the usual way by introducing ray path coordinates, i.e., auxilliary functions $q_\tau(\sigma)$, $q_\varphi(\sigma)$, $q_\lambda(\sigma)$ and their conjugate momenta $p_\tau(\sigma)$, $p_\varphi(\sigma)$, $p_\lambda(\sigma)$ of a variable σ defined on an interval, $\sigma_0 \le \sigma \le \sigma_1$, that satisfy the end-point conditions

$$q_\tau(\sigma_0) = 0 \quad ; \quad q_\tau(\sigma_1) = \tau \tag{69}$$

$$q_\varphi(\sigma_0) = \varphi_0 \quad ; \quad q_\varphi(\sigma_1) = \varphi \tag{70}$$

$$q_\lambda(\sigma_0) = \lambda_0 \quad ; \quad q_\lambda(\sigma_1) = \lambda \tag{71}$$

$$p_\tau(\sigma_1) = \frac{\partial A}{\partial\tau} \tag{72}$$

$$p_\varphi(\sigma_1) = \frac{\partial A}{\partial\varphi} \tag{73}$$

$$p_\lambda(\sigma_1) = \frac{\partial A}{\partial \lambda} \tag{74}$$

The initial values for the momenta are picked so that $\mathcal{E}(\sigma_0) = 0$ where

$$\mathcal{E}(\sigma) \equiv p_\tau(\sigma) + \frac{1}{2}\frac{1}{h_\varphi^2(\sigma)}(p_\varphi(\sigma))^2 + \frac{1}{2}\frac{1}{h_\lambda^2(\sigma)}(p_\lambda(\sigma))^2 \tag{75}$$

is the Hamiltonian function. Here we introduce the notation

$$h_\varphi(\sigma) \equiv \mu(q_\varphi(\sigma)) \; ; \; h_\lambda(\sigma) \equiv \nu(q_\varphi(\sigma)) \cos(q_\varphi(\sigma)) \tag{76}$$

The eikonal equation is equivalent to $\mathcal{E}(\sigma_1) = 0$ and will be satisfied provided

$$\frac{d\mathcal{E}(\sigma)}{d\sigma} = 0$$

This condition, in turn, is assured if the path variables satisfy Hamilton's equations

$$\frac{dq_\tau}{d\sigma} = 1 \tag{77}$$

$$\frac{dq_\varphi}{d\sigma} = \frac{1}{h_\varphi^2} p_\varphi \tag{78}$$

$$\frac{dq_\lambda}{d\sigma} = \frac{1}{h_\lambda^2} p_\lambda \tag{79}$$

and

$$\frac{dp_\tau}{d\sigma} = 0 \tag{80}$$

$$\frac{dp_\varphi}{d\sigma} = -\frac{1}{2}\left[\frac{\partial}{\partial q_\varphi}\left(\frac{1}{h_\varphi^2}\right) p_\varphi^2 + \frac{\partial}{\partial q_\varphi}\left(\frac{1}{h_\lambda^2}\right) p_\lambda^2\right] \tag{81}$$

$$\frac{dp_\lambda}{d\sigma} = 0 \tag{82}$$

From Eqs.(69) and (77) we have

$$q_\tau(\sigma) = \sigma - \sigma_0 \quad ; \quad \sigma_1 - \sigma_0 = \tau \tag{83}$$

In general, the eikonal A is given by the integral

$$A = \int_{\sigma_0}^{\sigma_1} d\sigma \left[p_\tau \frac{dq_\tau}{d\sigma} + p_\varphi \frac{dq_\varphi}{d\sigma} + p_\lambda \frac{dq_\lambda}{d\sigma}\right] \tag{84}$$

According to Eqs.(77)-(80), this integral can be rewritten as

$$A = \tau \, p_\tau + \int_{\sigma_0}^{\sigma_1} \mathbf{v}^2 d\sigma \tag{85}$$

where

$$\mathbf{v} = \frac{ds}{d\sigma} = \sqrt{\frac{1}{h_\varphi^2}p_\varphi^2 + \frac{1}{h_\lambda^2}p_\lambda^2} \tag{86}$$

Because $\mathcal{E} = 0$,

$$-2p_\tau = \mathbf{v}^2 = \text{constant}$$

Therefore

$$A = -\tau\ p_\tau \tag{87}$$

Consider now the path length

$$s(\sigma) = \int_{\sigma_0}^{\sigma} ds = \int_{\sigma_0}^{\sigma} \mathbf{v}(\sigma')d\sigma' = (\sigma - \sigma_0)\sqrt{-2\ p_\tau} \tag{88}$$

Equation (87) becomes

$$A = \frac{S^2}{2\tau} \tag{89}$$

Here $S = s(\sigma_1)$ is the total path length along the ray path. This remarkable expression gives the functional dependence of A on τ since S is independent of τ.

4.3 Derivatives along the ray path

Consider a function $f = f(q_\tau, q_\varphi, q_\lambda, \zeta)$ of the depth variable ζ and the path variables. The derivative of f along the path is

$$\begin{aligned}\frac{df}{d\sigma} &= \frac{dq_\tau}{d\sigma}\frac{\partial f}{\partial q_\tau} + \frac{dq_\varphi}{d\sigma}\frac{\partial f}{\partial q_\varphi} + \frac{dq_\lambda}{d\sigma}\frac{\partial f}{\partial q_\lambda} \\ &= \frac{\partial f}{\partial q_\tau} + \frac{1}{h_\varphi^2}p_\varphi\frac{\partial f}{\partial q_\varphi} + \frac{1}{h_\lambda^2}p_\lambda\frac{\partial f}{\partial q_\lambda}\end{aligned}$$

If this expression is evaluated at the end point $\sigma = \sigma_1$, we have

$$\frac{df}{d\sigma}\Big|_{\sigma=\sigma_1} = \frac{\partial f(\tau,\varphi,\lambda,\zeta)}{\partial\tau} + \frac{1}{\mu^2}\frac{\partial A}{\partial\varphi}\frac{\partial f(\tau,\varphi,\lambda,\zeta)}{\partial\varphi} + \frac{1}{\nu^2\cos^2\varphi}\frac{\partial A}{\partial\lambda}\frac{\partial f(\tau,\varphi,\lambda,\zeta)}{\partial\lambda} \tag{90}$$

One can change the independent variable in the ray paths from σ to the path length s since $\mathbf{v}(s) = ds/d\sigma$ does not change sign. It is, in fact, a constant equal to S/τ. Equation (88) is first solved for σ in terms of s,

$$\sigma(s) = \sigma_0 + \int_0^s \frac{1}{\mathbf{v}}ds' = \sigma_0 + s\frac{\tau}{S} \tag{91}$$

and new ray path coordinates are introduced

$$\tau_{ray}(s) \equiv q_\tau(\sigma(s))$$

$$\varphi_{ray}(s) \equiv q_\varphi(\sigma(s))$$
$$\lambda_{ray}(s) \equiv q_\lambda(\sigma(s))$$

defined on the interval, $0 \leq s \leq S$, that satisfy the same end-point conditions satisfied by q_τ, q_φ, and q_λ. In terms of these new path variables, a function $f(s,\zeta)) = f(\tau_{ray}(s), \varphi_{ray}(s), \lambda_{ray}(s), \zeta)$ has a derivative with respect to path length given by

$$\mathbf{v}\frac{\partial f}{\partial s} = \left[\frac{dq_\tau}{d\sigma}\frac{\partial f}{\partial \tau_{ray}} + \frac{dq_\varphi}{d\sigma}\frac{\partial f}{\partial \varphi_{ray}} + \frac{dq_\lambda}{d\sigma}\frac{\partial f}{\partial \lambda_{ray}}\right]$$

where the derivatives with respect to σ are evaluated at $\sigma(s)$. At the end point of the path this expression becomes

$$\frac{S}{\tau}\left[\frac{\partial f}{\partial s}\right]_{s=S} = \frac{\partial f(\tau,\varphi,\lambda,\zeta)}{\partial \tau} + \frac{1}{\mu^2}\frac{\partial A}{\partial \varphi}\frac{\partial f(\tau,\varphi,\lambda,\zeta)}{\partial \varphi} + \frac{1}{\nu^2\cos^2\varphi}\frac{\partial A}{\partial \lambda}\frac{\partial f(\tau,\varphi,\lambda,\zeta)}{\partial \lambda} \quad (92)$$

We have reverted to using a partial derivative on the left-hand-side of this equation to remind ourselves f can be a function of depth as well as of the path variables. By making this change in the independent variable, we will end up with equations that, unlike Eqs.(59) and (129) (below), involve τ as a parameter rather than as an independent variable thus simplifying the evaluation of Eq.(53).

The unit vector along the ray path can be constructed from Eq.(33)

$$\hat{s}(\sigma) = \left(h_\varphi(\sigma)\,\frac{dq_\varphi(\sigma)}{d\sigma}\,\hat{\varphi} + h_\lambda(\sigma)\,\frac{dq_\lambda(\sigma)}{d\sigma}\,\hat{\lambda}\right)/\frac{ds}{d\sigma} \quad (93)$$

which at the end point of the path becomes

$$\frac{S}{\tau}\hat{s}(\sigma_1) = \frac{1}{\mu}\frac{\partial A}{\partial \varphi}\,\hat{\varphi} + \frac{1}{\nu\cos\varphi}\frac{\partial A}{\partial \lambda}\,\hat{\lambda} \quad (94)$$

4.4 Solving the transport equation

If we make the identification in Eq.(90)

$$f(\sigma_1) = f(\tau,\varphi,\lambda) = \ln \hat{H}^2$$

Eq.(67) becomes

$$\frac{df}{d\sigma}|_{\sigma=\sigma_1} + \nabla_S^2 A = 0 \quad (95)$$

For this expression to be useful, we must write $\nabla_S^2 A$ in terms of path variables evaluated at the end point $\sigma = \sigma_1$. To this end we define

$$A_\varphi(\sigma) \equiv \frac{h_\lambda(\sigma)}{h_\varphi(\sigma)}p_\varphi(\sigma)$$

and

$$A_\lambda(\sigma) \equiv \frac{h_\varphi(\sigma)}{h_\lambda(\sigma)} p_\lambda(\sigma)$$

So that

$$\nabla_S^2 A = \frac{1}{J(\varphi)} \left[\frac{\partial A_\varphi(\sigma_1)}{\partial \varphi} + \frac{\partial A_\lambda(\sigma_1)}{\partial \lambda} \right]$$

Because $\nabla_S^2 A$ involves derivatives with respect to φ and λ, the end-point values of the path coordinates q_φ and q_λ, it is clear it cannot be written as a function of the path (evaluated at the end point). What is needed, in addition, is a perturbed path whose coordinates differ at the end point from φ and λ by infinitesimal amounts $\delta\varphi$ and $\delta\lambda$. The perturbations δA_φ and δA_λ evaluated at the end points, will involve derivatives of A_φ and A_λ with respect to φ and λ. These derivatives can be used to construct $\nabla_S^2 A$. In addition, since $\nabla_S^2 A$ does not involve a derivative with respect to τ, it can be held fixed.

There is some freedom in the choice of how the path is perturbed. The perturbation can be induced by changes in any set of parameters that, together with φ_0 and λ_0, characterize the ray. Here we will use a set that exploits the fact that p_τ and p_λ are constant along the ray path. Motivated by consideration of the exact ray-path solution for a spere, we define

$$W \equiv \sqrt{-2\, p_\tau} \quad ; \quad P = \frac{p_\lambda}{W} \tag{96}$$

The derivatives of the paths can be written in terms of W and P as follows:

$$\frac{dq_\varphi}{d\sigma} = \frac{A_\varphi}{J} = \frac{W}{J}\sqrt{h_\lambda^2 - P^2} \quad ; \quad \frac{dq_\lambda}{d\sigma} = \frac{A_\lambda}{J} = \frac{WP}{h_\lambda^2} \tag{97}$$

Now $A_\varphi(\sigma_1)$ and $A_\lambda(\sigma_1)$ depend on φ and λ (as well as τ) so we can write the matrix equation

$$\begin{bmatrix} \partial A_\varphi(\sigma_1)/\partial W & \partial A_\varphi(\sigma_1)/\partial P \\ \partial A_\lambda(\sigma_1)/\partial W & \partial A_\lambda(\sigma_1)/\partial P \end{bmatrix} = \begin{bmatrix} \partial A_\varphi(\sigma_1)/\partial \varphi & \partial A_\varphi(\sigma_1)/\partial \lambda \\ \partial A_\lambda(\sigma_1)/\partial \varphi & \partial A_\lambda(\sigma_1)/\partial \lambda \end{bmatrix} Q\,(\sigma_1) \tag{98}$$

where

$$Q\,(\sigma) \equiv \begin{bmatrix} \partial q_\varphi(\sigma)/\partial W & \partial q_\varphi(\sigma)/\partial P \\ \partial q_\lambda(\sigma)/\partial W & \partial q_\lambda(\sigma)/\partial P \end{bmatrix} \tag{99}$$

Equation (98) can be inverted provided that the determinant of Q,

$$\det(Q(\sigma)) = \frac{\partial q_\varphi(\sigma)}{\partial W}\frac{\partial q_\lambda(\sigma)}{\partial P} - \frac{\partial q_\varphi(\sigma)}{\partial P}\frac{\partial q_\lambda(\sigma)}{\partial W} \tag{100}$$

does not vanish at $\sigma = \sigma_1$. (Caustics are located where $\det Q = 0$.) By taking the trace after inverting Eq.(98) we obtain

$$\nabla_S^2 A = \frac{1}{J(\varphi)} trace \left\{ \begin{bmatrix} \partial A_\varphi(\sigma_1)/\partial \varphi & \partial A_\varphi(\sigma_1)/\partial \lambda \\ \partial A_\lambda(\sigma_1)/\partial \varphi & \partial A_\lambda(\sigma_1)/\partial \lambda \end{bmatrix} \right\}$$

$$= \frac{d \ln D(\sigma)}{d\sigma}\Big|_{\sigma=\sigma_1} \tag{101}$$

where

$$D(\sigma) \equiv J(q_\varphi(\sigma)) \det(Q(\sigma)) \tag{102}$$

This result was arrived at by using the relation

$$\frac{dD}{d\sigma} = \frac{\partial A_\varphi}{\partial W}\frac{\partial q_\lambda}{\partial P} - \frac{\partial A_\lambda}{\partial W}\frac{\partial q_\varphi}{\partial P} - (W \longleftrightarrow P) \tag{103}$$

which follows from the equations of motion. Except for the presence of the function $J(q_\varphi)$, the derivation of Eq.(101) is identical to the usual derivation for the Cartesian problem using Liousville's formula (see, e.g., Brekhovskikh and Godin, 1999, Sec. 5.1).

Combining Eq.(101) with Eq.(95) gives

$$\frac{df}{d\sigma}\Big|_{\sigma=\sigma_1} + \frac{d}{d\sigma} \ln D(\sigma)\Big|_{\sigma=\sigma_1} = 0$$

This equation can be solved by solving the more general one

$$\frac{df(\sigma)}{d\sigma} + \frac{d}{d\sigma} \ln D(\sigma) = 0$$

and evaluating the result at $\sigma = \sigma_1$,

$$\hat{H} = \mathcal{C}\sqrt{\frac{1}{D(\sigma_1)}}$$

where $\mathcal{C}$ is an integration constant. It can be determined by considering the solution to the parabolic equation at a point $(\varphi', \lambda', \tau')$ where φ' and λ' are located in a neighborhood of the source at φ_0 and λ_0 taken small enough so that μ and $\nu \cos\varphi$ can be considered constant within it. The solution to the horizontal parabolic equation at $(\varphi', \lambda', \tau')$ that satisfies Eq.(60) is simply

$$H(\tau'; \varphi', \lambda') = \left(\frac{k_0}{2\pi i \tau'}\right) \exp\left\{\frac{ik_0}{2\tau'}\left[\mu_0^2 (\varphi' - \varphi_0)^2 + \nu_0^2 \cos^2\varphi_0 (\lambda' - \lambda_0)^2\right]\right\}$$

where $\mu_0 = \mu(\varphi_0)$ and $\nu_0 = \nu(\varphi_0)$. We take the point $(\varphi', \lambda', \tau')$ to be along the ray path from $(\varphi_0, \lambda_0, 0)$ to (φ, λ, τ);

$$\varphi' = q_\varphi(\sigma') \quad ; \quad \lambda' = q_\lambda(\sigma') \quad ; \quad \tau' = \sigma' - \sigma_0$$

The variable $\sigma' - \sigma_0 \lesssim \tau'$ is necessarily small enough so that the ray paths can be approximated by straight lines,

$$q_\varphi(\sigma') - \varphi_0 = (\sigma' - \sigma_0)\frac{dq_\varphi}{d\sigma}(\sigma_0) = (\sigma' - \sigma_0)\frac{W}{J(\varphi_0)}\sqrt{\nu_0^2 \cos^2\varphi_0 - P^2}$$

$$q_\lambda(\sigma') - \lambda_0 = (\sigma' - \sigma_0)\frac{dq_\lambda}{d\sigma}(\sigma_0) = (\sigma' - \sigma_0)\frac{WP}{\nu_0^2 \cos^2\varphi_0}$$

It is clear from these expressions that $D(\sigma')$ can be written in the form

$$D(\sigma') = \frac{1}{2}(\sigma' - \sigma_0)^2 \frac{d^2 D}{d\sigma^2}(\sigma_0) \tag{104}$$

(The second derivative in this expression is, in fact, equal to $2W^2/A_\varphi(\sigma_0)$.) Since

$$\hat{H}(\sigma') = \frac{k_0}{2\pi i\,(\sigma' - \sigma_0)}$$

we have

$$\mathcal{C} = \hat{H}(\sigma')\sqrt{D(\sigma')} = \frac{k_0}{2\pi i}\sqrt{\frac{1}{2}\frac{d^2 D}{d\sigma^2}(\sigma_0)}$$

The amplitude factor is, therefore,

$$\hat{H} = \frac{k_0}{2\pi i}\sqrt{\frac{1}{2D(\sigma_1)}\frac{d^2 D}{d\sigma^2}(\sigma_0)}$$

4.5 Calculating $D(\sigma)$

If both sides of Eqs.(97) are differentiated with respect to W and P, we obtain the relations

$$\frac{\partial A_\varphi}{\partial W} = \frac{A_\varphi}{W} + J\Omega\frac{\partial q_\varphi}{\partial W} \tag{105}$$

$$\frac{\partial A_\varphi}{\partial P} = -\frac{W^2 P}{A_\varphi} + J\Omega\frac{\partial q_\varphi}{\partial P} \tag{106}$$

$$\frac{\partial A_\lambda}{\partial W} = \frac{A_\lambda}{W} + \frac{\partial A_\lambda}{\partial q_\varphi}\frac{\partial q_\varphi}{\partial W} \tag{107}$$

$$\frac{\partial A_\lambda}{\partial P} = \frac{A_\lambda}{P} + \frac{\partial A_\lambda}{\partial q_\varphi}\frac{\partial q_\varphi}{\partial P} \tag{108}$$

with

$$\Omega = \frac{1}{J}\frac{\partial A_\varphi}{\partial q_\varphi} = \frac{W^2}{2A_\varphi}\frac{\partial h_\lambda^2}{\partial q_\varphi} \tag{109}$$

and

$$\frac{\partial A_\lambda}{\partial q_\varphi} = A_\lambda \frac{\partial}{\partial q_\varphi}\ln\left(\frac{J}{h_\lambda^2}\right) \tag{110}$$

Substituting these expressions into Eq.(103) yields

$$\frac{d}{d\sigma}\left(\frac{D}{A_\varphi}\right) = \frac{D_1}{A_\varphi} + \frac{D_2}{A_\varphi} \tag{111}$$

where

$$D_1 = \frac{1}{W}\left(\frac{W^2\left(h_\lambda^2 - P^2\right)}{A_\varphi}\frac{\partial q_\lambda}{\partial P} - A_\lambda\frac{\partial q_\varphi}{\partial P}\right) \tag{112}$$

and

$$D_2 = \frac{1}{P}\left(\frac{W^2P^2}{A_\varphi}\frac{\partial q_\lambda}{\partial W} + A_\lambda\frac{\partial q_\varphi}{\partial W}\right) \tag{113}$$

The derivatives of D_1 and D_2 with respect to σ can be calculated by using again Eqs.(97) and (105)-(108) with the result

$$\frac{d}{d\sigma}\left(\frac{D_1}{A_\varphi}\right) = \frac{W^2}{A_\varphi^2} \tag{114}$$

and

$$\frac{d}{d\sigma}\left(A_\varphi D_2\right) = W \tag{115}$$

Eq.(115) can be solved for D_2:

$$D_2 = (\sigma - \sigma_0)\frac{W^2}{A_\varphi} \tag{116}$$

Using Eqs.(114) and (116) one sees that $(\sigma - \sigma_0)D_1$ obeys the same equation obeyed by D (Eq.(111)). Since D and $(\sigma - \sigma_0)D_1$ satisfy the same initial conditions, we have

$$D(\sigma) = (\sigma - \sigma_0)D_1(\sigma) \tag{117}$$

Differentiating both sides of Eq.(114) gives

$$\frac{d^2D_1}{d\sigma^2} - \left(\frac{d\Omega}{d\sigma} + \Omega^2\right)D_1 = 0 \tag{118}$$

The term in parentheses can be written in a particularly convenient form by differentiating Ω using the relations $dA_\varphi/d\sigma = -W^2\sin(q_\varphi)$ and $\partial h_\lambda^2/\partial q_\varphi = -2\,J\,\sin(q_\varphi)$, which follow from Eqs.(97) and (31). One finds

$$-\left(\frac{d\Omega}{d\sigma} + \Omega^2\right) = \frac{W^2}{\mu\nu} > 0 \tag{119}$$

Referring to Eq.(35), we can write Eq.(118) as

$$\frac{d^2D_1}{d\sigma^2} + \frac{W^2}{R^2}D_1 = 0 \tag{120}$$

If the Earth is modeled as a sphere, both μ and ν would be equal to its radius, $\bar{R}$, and the solution for D_1 is immediately found without approximation to be

$$D_1(\sigma) = \frac{1}{2}\frac{d^2D}{d\sigma^2}(\sigma_0)\frac{\bar{R}}{W}\sin\left[\frac{W}{\bar{R}}(\sigma - \sigma_0)\right]$$

At the end of the path the argument of the sine function becomes $S/\bar{R}$ which is easily seen to be the angle at the center of the Earth subtended by the arc that ends at source and receiver positions

$$\cos(\frac{S}{\bar{R}}) = \cos\varphi\cos\varphi_0\cos(\lambda-\lambda_0) + \sin\varphi\sin\varphi_0$$

Equation (120) cannot be solved exactly for an arbitrary volume of revolution. However, if

$$\frac{1}{W}\left|\frac{dR}{d\sigma}\right| \ll 1 \tag{121}$$

we can approximate D_1 with its WKB solution

$$D_1(\sigma) \approx D_1^{WKB}(\sigma) = \frac{1}{2W}\frac{d^2D}{d\sigma^2}(\sigma_0)\sqrt{R(\sigma)R(\sigma_0)}\sin\left(W\int_{\sigma_0}^{\sigma}\frac{d\sigma'}{R(\sigma')}\right) \tag{122}$$

The overall normalization has been determined using Eq.(104).

To explore the validity of the condition (121), we use the parametrization Eq.(36) to find

$$\frac{dR}{d\sigma} = -\frac{\bar{R}\eta}{2\left(1+\eta g(q_\varphi)\right)^{3/2}}\frac{\partial g(q_\varphi)}{\partial q_\varphi}\frac{dq_\varphi}{d\sigma}$$

Since

$$\left|\frac{dq_\varphi}{d\sigma}\right| \leq \frac{W}{h_\varphi} \approx \frac{W}{\bar{R}}$$

we have

$$\frac{1}{W}\left|\frac{dR}{d\sigma}\right| \lesssim \frac{1}{2}\eta\left|\frac{\partial g(q_\varphi)}{\partial q_\varphi}\right| \tag{123}$$

The condition (121) will be satisfied if the smallest scale of variability in latitude of the surface is larger than the deviation of the Earth's shape from that of a sphere. If we (arbitrarily) require that the right-hand-side of Eq.(123) to be less than 1/10, the condition is satisfied provided one can consider the Earth spherical on horizontal scales of order $10\left(\bar{R}\eta/2\right) \approx 210$ km. Clearly Eq.(121) is valid since over distances of a few hundred kilometers it is safe to assume the Earth is flat–one does not need to even assume it is spherical.

Collecting expressions gives

$$\hat{H} = \frac{k_0\sqrt{W}}{2\pi i\left(R(\sigma_1)R(\sigma_0)\right)^{1/4}}\left[(\sigma_1-\sigma_0)\sin\left(W\int_{\sigma_0}^{\sigma_1}\frac{d\sigma'}{R(\sigma')}\right)\right]^{-1/2}$$

Changing the independent variable from σ to s and introducing the path-averaged quantity

$$\frac{1}{R_{ave}} = \frac{1}{S}\int_0^S \frac{ds'}{R(s')} = \frac{1}{S}\int_0^S \frac{ds'}{\sqrt{\mu(\varphi_{ray}(s'))\nu(\varphi_{ray}(s'))}} \tag{124}$$

results in the expression

$$\hat{H} = \frac{k_0}{2\pi i\tau}\left(\frac{R_{ave}}{R(S)}\frac{R_{ave}}{R(0)}\right)^{1/4}\left[\frac{S/R_{ave}}{\sin\left(S/R_{ave}\right)}\right]^{1/2} \tag{125}$$

The ray acoustics approximation to Eq.(59) is therefore

$$H \simeq H_{ray} = \frac{k_0}{2\pi i\tau}\left(\frac{R_{ave}}{R(S)}\frac{R_{ave}}{R(0)}\right)^{1/4}\left[\frac{S/R_{ave}}{\sin\left(S/R_{ave}\right)}\right]^{1/2} \exp(\frac{ik_0}{2\tau}S^2) \tag{126}$$

5. THE VERTICAL EQUATION AND STATIONARY-PHASE

The horizontal ray acoustics approximation for Φ is slightly more involved than for H because no assumptions are made about the magnitude of the scales of variability of the depth coordinate. For Eqs.(56) and (59) to be satisfied, Ψ must satisfy the matrix equation

$$\begin{bmatrix} 2ik_0J\left(ik_0V + \partial/\partial\tau\right) & 2ik_0\partial A/\partial\varphi + \partial/\partial\varphi & 2ik_0\partial A/\partial\lambda + \partial/\partial\lambda & J\rho\partial/\partial\zeta \\ \partial/\partial\varphi & -\mu/\left(\nu\cos\varphi\right) & 0 & 0 \\ \partial/\partial\lambda & 0 & -\left(\nu\cos\varphi\right)/\mu & 0 \\ J\rho\partial/\partial\zeta & 0 & 0 & -J\rho^2 \end{bmatrix}$$

$$\times \begin{bmatrix} \Psi \\ \Psi_\varphi \\ \Psi_\lambda \\ \Psi_\zeta \end{bmatrix} = 0 \tag{127}$$

and the initial condition

$$Lim_{\tau\to 0}\Psi(\tau;\varphi,\lambda,\zeta) = \delta\left(\zeta - \zeta_0\right) \tag{128}$$

Just as before we drop the horizontal gradients of the amplitude factors in comparison to the corresponding gradients of A, multiplied by k_0. While the derivative $\partial/\partial\tau$ can be dropped in the matrix in Eq.(63) we are not justified in dropping it in Eq.(127). The result is

$$\begin{bmatrix} 2ik_0J\left(ik_0V + \partial/\partial\tau\right) & 2ik_0\partial A/\partial\varphi & 2ik_0\partial A/\partial\lambda & J\rho\partial/\partial\zeta \\ \partial/\partial\varphi & -\mu/\left(\nu\cos\varphi\right) & 0 & 0 \\ \partial/\partial\lambda & 0 & -\left(\nu\cos\varphi\right)/\mu & 0 \\ J\rho\partial/\partial\zeta & 0 & 0 & -J\rho^2 \end{bmatrix}$$

$$
\times \begin{bmatrix} \Psi \\ \Psi_\varphi \\ \Psi_\lambda \\ \Psi_\zeta \end{bmatrix} = 0
$$

which is equivalent to the single second-order equation

$$
\left[2ik_0\left(\frac{\partial}{\partial\tau}+\frac{1}{\mu^2}\frac{\partial A}{\partial\varphi}\frac{\partial}{\partial\varphi}+\frac{1}{\nu^2\cos^2\varphi}\frac{\partial A}{\partial\lambda}\frac{\partial}{\partial\lambda}\right)+\rho\frac{\partial}{\partial\zeta}\left(\frac{1}{\rho}\frac{\partial}{\partial\zeta}\right)-2k_0^2V\right]\Psi=0 \quad (129)
$$

Comparison of Eqs.(129) and (92) gives

$$
2ik_0\frac{S}{\tau}\left[\frac{\partial\Psi(\tau;\varphi_{ray}(s),\lambda_{ray}(s),\zeta)}{\partial s}\right]_{s=S}
$$

$$
+\left[\rho\frac{\partial}{\partial\zeta}\left(\frac{1}{\rho}\frac{\partial}{\partial\zeta}\right)-2k_0^2V(\varphi_{ray}(S),\lambda_{ray}(S),\zeta)\right]\Psi(\tau;\varphi_{ray}(S),\lambda_{ray}(S),\zeta)=0
$$

This equation can be solved by solving the more general one

$$
\left[2ik_0\frac{S}{\tau}\frac{\partial}{\partial s}+\rho\frac{\partial}{\partial\zeta}\left(\frac{1}{\rho}\frac{\partial}{\partial\zeta}\right)-2k_0^2V(\varphi_{ray}(s),\lambda_{ray}(s),\zeta)\right]\Psi(\tau;\varphi_{ray}(s),\lambda_{ray}(s),\zeta)=0 \quad (130)
$$

and evaluating the result at $s = S$.

If we require that

$$
Lim_{s\to 0}\Psi(\tau;\varphi_{ray}(s),\lambda_{ray}(s),\zeta)=\delta(\zeta-\zeta_0) \quad (131)
$$

then the initial condition, Eq.(128), will be satisfied because of the relation (91). In fact, independent of the ray acoustics approximation, it is not difficult to see that Ψ has support only in a small neighborhood of ζ about ζ_0 and τ about 0 as $s \to 0$.

Collecting expressions and substituting into Eq.(53) give

$$
G(\vec{x}\,|\,\vec{x}_0)\approx\frac{1}{4\pi}\left(\frac{R_{ave}}{R(S)}\frac{R_{ave}}{R(0)}\right)^{1/4}\left[\frac{S/R_{ave}}{\sin(S/R_{ave})}\right]^{1/2}
$$

$$
\times\int_0^\infty\frac{d\tau}{\tau}\exp\frac{ik_0}{2}(\tau+\frac{S^2}{\tau})\Psi(\tau;\varphi,\lambda,\zeta) \quad (132)
$$

One can evaluate the integral over the Fock-Feynman parameter in Eq.(132) using the method of stationary phase. The single stationary phase point is at $\tau = S$. If $\Psi(\tau;\varphi,\lambda,\zeta)$ has a phase which is a slowly varying function of τ at $\tau = S$, justifying the stationary phase approximation, then the wave number spectrum of $\Psi(S;\varphi,\lambda,\zeta)$ in path length—the horizontal wave number spectrum—will be dominated by values near the reference wave number k_0. That is, the stationary-phase approximation is equivalent to the narrow-angle approximation (Palmer, 1976).

The stationary phase approximation yields

$$G(\vec{x}\,|\,\vec{x}_0) \approx \frac{1}{4\pi}\left(\frac{R_{ave}}{R(S)}\frac{R_{ave}}{R(0)}\right)^{1/4}\left[\frac{2\pi i}{k_0 R_{ave}\sin\left(S/R_{ave}\right)}\right]^{1/2} e^{ik_0 S}\,\psi(S,\zeta) \tag{133}$$

where

$$\psi(s,\zeta) \equiv \Psi(S;\varphi_{ray}(s),\lambda_{ray}(s),\zeta)$$

satisifies the two-dimensional parabolic equation

$$\left[2ik_0\frac{\partial}{\partial s} + \rho\frac{\partial}{\partial\zeta}\left(\frac{1}{\rho}\frac{\partial}{\partial\zeta}\right) - 2k_0^2 V(s,\zeta)\right]\Psi(s,\zeta) = 0 \tag{134}$$

and the initial condition

$$Lim_{s\to 0}\psi(s,\zeta) = \delta\left(\zeta - \zeta_0\right) \tag{135}$$

It being understood that s is measured along the ray path and that

$$V(s,\zeta) = V(\varphi_{ray}(s),\lambda_{ray}(s),\zeta) \tag{136}$$

If the Earth is modeled as a sphere with radius $\bar{R}$, Eq.(133) reduces to

$$G(\vec{x}\,|\,\vec{x}_0) \approx \frac{1}{4\pi}\left[\frac{2\pi i}{k_0\bar{R}\sin\left(S/\bar{R}\right)}\right]^{1/2} e^{ik_0 S}\,\psi(S,\zeta) \qquad \text{(sphere)}$$

It is worth noting that $\psi(s,\zeta)$ does not depend on the total path length S because $\Psi(\tau;\varphi_{ray}(s),\lambda_{ray}(s),\zeta)$ depends on S and τ only through the ratio S/τ. Eq.(134) is clearly energy conserving and can be solved numerically using a standard range-sliced marching algorithm. Moreover, it does not possess any kinematic singularities of the type experienced by Collins *et al.*, (1996). Caustics in the horizontal are located at $S \approx \pi R_{ave}$ and correspond to the focusing that occurs for near antipodal propagation.

6. REFRACTED GEODESICS AND NORMAL MODES

In this section we consider two extensions of the method we have presented.

6.1 Refracted Geodesics.

The factorization of Φ, Eq.(58), into a term H that depends on the horizontal coordinates and a remainder is not unique. One can always define H by the equation

$$\left[2ik_0\frac{\partial}{\partial\tau} + \nabla_\perp^2 - 2k_0^2 U(\varphi,\lambda)\right] H(\tau;\varphi,\lambda) = 0 \tag{137}$$

where

$$U(\varphi, \lambda) = -\frac{1}{2}\left[n_{refr}^2(\varphi, \lambda) - 1\right] \tag{138}$$

is a function of φ and λ but not of the depth coordinate ζ. In Eq.(127) and all subsequent equations $V(\varphi, \lambda, \zeta)$ would be replaced with $\tilde{V}(\varphi, \lambda, \zeta)$ where

$$\tilde{V}(\varphi, \lambda, \zeta) = V(\varphi, \lambda, \zeta) - U(\varphi, \lambda) \tag{139}$$

We want to indicate how the analysis presented above is modified by the presence of the function U.

The horizontal ray-acoustics approximation for H now involves a ray path that is a refracted geodesic. The eikonal equation and the expression for the Hamiltonian function become, respectively,

$$\frac{\partial A}{\partial \tau} + \frac{1}{2\mu^2}\left(\frac{\partial A}{\partial \varphi}\right)^2 + \frac{1}{2\nu^2 \cos^2 \varphi}\left(\frac{\partial A}{\partial \lambda}\right)^2 + U = 0 \tag{140}$$

and

$$\mathcal{E} = p_\tau + \frac{1}{2}\mathbf{v}^2 + U = 0 \tag{141}$$

where $\mathbf{v}$ is now no longer a constant. The equations for p_φ and p_λ become

$$\frac{dp_\varphi}{d\sigma} = -\frac{1}{2}\left[\frac{\partial}{\partial q_\varphi}\left(\frac{1}{h_\varphi^2}\right)p_\varphi^2 + \frac{\partial}{\partial q_\varphi}\left(\frac{1}{h_\lambda^2}\right)p_\lambda^2\right] - \frac{\partial U}{\partial q_\varphi}$$

and

$$\frac{dp_\lambda}{d\sigma} = -\frac{\partial U}{\partial q_\lambda}$$

The equations for all the other ray-path variables are unchanged. The eikonal is now

$$A = \tau p_\tau + \tilde{A} \tag{142}$$

where $\tilde{A}$ is the horizontal eikonal function

$$\tilde{A} = \int_{\sigma_0}^{\sigma_1} \mathbf{v}^2 d\sigma = \int_0^s \mathbf{v} ds \tag{143}$$

Since Eq.(142) can be written as

$$A - \tau\frac{\partial A}{\partial \tau} = \tilde{A} \tag{144}$$

it defines a Legendre transform which can be used to change the independent variables from (φ, λ, τ) to $(\varphi, \lambda, p_\tau)$. The variable τ is then given by

$$\tau = -\frac{\partial \tilde{A}}{\partial p_\tau} = \int_0^s \frac{ds}{\sqrt{(-2)\left(p_\tau + U(s)\right)}}$$

The exponent in Eq.(132) is now

$$ik_0(\frac{\tau}{2}+A)$$

instead of $ik_0(\tau+S^2/\tau)/2$. At the stationary-phase point $\tau=\tau_{s.p.}$ we have

$$\frac{\partial A}{\partial \tau}=p_\tau=-\frac{1}{2} \tag{145}$$

so

$$\mathbf{v}|_{\tau=\tau_{s.p.}}=\sqrt{1-2U}=n_{refr}$$

and

$$\tau_{s.p.}=\int_0^S \frac{ds}{n_{refr}}$$

We then have

$$ik_0(\frac{\tau}{2}+A)_{\tau=\tau_{s.p.}}=ik_0\tilde{A}_{\tau=\tau_{s.p.}}=ik_0\int_0^S n_{refr}ds$$

The stationary-phase approximation then gives

$$G(\vec{x}\,|\,\vec{x}_0)\propto\int_0^\infty \hat{H}\exp\left[ik_0(\frac{\tau}{2}+A)\right]\Psi(\tau;\varphi,\lambda,\zeta)$$

$$\propto\left[\hat{H}\left(\frac{\partial^2 A}{\partial\tau^2}\right)^{-1/2}\right]_{\tau=\tau_{s.p.}}\exp\left[ik_0\int_0^S n_{refr}ds\right]\Psi(\tau_{s.p.};\varphi,\lambda,\zeta) \tag{146}$$

In calculating $\hat{H}$ it is no longer useful to use the set W and P to characterize ray-path variations. Instead we use

$$v_\varphi\equiv p_\varphi(\sigma_0)=-\frac{\partial A}{\partial\varphi_0} \tag{147}$$

$$v_\lambda\equiv p_\lambda(\sigma_0)=-\frac{\partial A}{\partial\lambda_0} \tag{148}$$

The relationship between these initial values for the momenta and the derivatives of the eikonal with respect to the coordinates of the source position follow easily from the defining equation, Eq.(84). We still have

$$\hat{H}\propto\sqrt{\frac{1}{D(\sigma_1)}}$$

but now

$$D(\sigma)\equiv J(q_\varphi(\sigma))\det\begin{bmatrix}\partial q_\varphi(\sigma)/\partial v_\varphi & \partial q_\varphi(\sigma)/\partial v_\lambda\\ \partial q_\lambda(\sigma)/\partial v_\varphi & \partial q_\lambda(\sigma)/\partial v_\lambda\end{bmatrix}$$

We next consider the variation in the ray path due to a change in receiver position

$$\begin{bmatrix} \partial q_\varphi(\sigma)/\partial\varphi & \partial q_\varphi(\sigma)/\partial\lambda \\ \partial q_\lambda(\sigma)/\delta\varphi & \partial q_\lambda(\sigma)/\partial\lambda \end{bmatrix} = -O \times \begin{bmatrix} \partial q_\varphi(\sigma)/\partial v_\varphi & \partial q_\varphi(\sigma)/\partial v_\lambda \\ \partial q_\lambda(\sigma)/\partial v_\varphi & \partial q_\lambda(\sigma)/\partial v_\lambda \end{bmatrix}$$

where

$$O = \begin{bmatrix} \partial^2 A/\partial\varphi_0\partial\varphi & \partial^2 A/\partial\varphi_0\partial\lambda \\ \partial^2 A/\partial\lambda_0\partial\varphi & \partial^2 A/\partial\lambda_0\partial\lambda \end{bmatrix} \tag{149}$$

(We have used $\partial v_\varphi/\partial\varphi = -\partial^2 A/\partial\varphi_0\partial\varphi$,etc.) At $\sigma = \sigma_1$ the left-hand-side of this equation becomes equal to the identity matrix and we have

$$D(\sigma_1) = \frac{J(\varphi)}{\det O}$$

We now turn to consideration of $\partial^2 A/\partial\tau^2$ in Eq.(146). At $\sigma = \sigma_0$, Eq.(141) becomes

$$p_\tau + \frac{1}{2h_\varphi^2(\varphi_0)}v_\varphi^2 + \frac{1}{2h_\lambda^2(\varphi_0)}v_\lambda^2 + U(\varphi_0, \lambda_0) = 0$$

Differentiating with respect to τ gives

$$\frac{\partial^2 A}{\partial\tau^2} + \frac{\partial v_\varphi}{\partial\tau}\frac{dq_\varphi}{d\sigma}(\varphi_0) + \frac{\partial v_\lambda}{\partial\tau}\frac{dq_\lambda}{d\sigma}(\varphi_0) = 0$$

The change in the eikonal equation due to a change in source location, keeping the receiver location fixed, is given by

$$\frac{\partial^2 A}{\partial\varphi_0\partial\varphi}\frac{dq_\varphi}{d\sigma}(\varphi) + \frac{\partial^2 A}{\partial\varphi_0\partial\lambda}\frac{dq_\lambda}{d\sigma}(\varphi) = \frac{\partial v_\varphi}{\partial\tau}$$

and

$$\frac{\partial^2 A}{\partial\lambda_0\partial\varphi}\frac{dq_\varphi}{d\sigma}(\varphi) + \frac{\partial^2 A}{\partial\lambda_0\partial\lambda}\frac{dq_\lambda}{d\sigma}(\varphi) = \frac{\partial v_\lambda}{\partial\tau}$$

Combining these last three equations gives

$$\frac{\partial^2 A}{\partial\tau^2} = e\left(\sigma_0\right)^T \cdot O \cdot e\left(\sigma_1\right) \tag{150}$$

where

$$e\left(\sigma\right) = \begin{bmatrix} dq_\varphi\left(\sigma\right)/d\sigma \\ dq_\lambda\left(\sigma\right)/d\sigma \end{bmatrix}$$

Therefore

$$\left[\hat{H}\left(\frac{\partial^2 A}{\partial\tau^2}\right)^{-1/2}\right]_{\tau=\tau_{s.p.}} = \left(\frac{1}{J(\varphi)}\frac{\det O}{e\left(\sigma_0\right)\cdot O\cdot e\left(\sigma_1\right)}\right)^{1/2}_{\tau=\tau_{s.p.}} \tag{151}$$

(For an alternative way of writing this amplitude factor see Gutzwiller (1990, Eq.

(2.10)).)

The evaluation of the derivatives of A with respect to source and receiver coordinates that make up the components of O are done at the stationary-phase point

$$A_{\tau=\tau_{s.p.}} = \frac{1}{2}\int_0^S \left[\frac{2n_{refr}^2 - 1}{n_{refr}}\right] ds \approx -\frac{1}{2}\int_0^S \frac{ds}{n_{refr}} \tag{152}$$

The derivatives can be performed once Hamilton's equations have been solved for the refracted geodesic and the variation of n_{refr} along the path is determined. Collecting expressions gives

$$G(\vec{x}\,|\,\vec{x}_0) \propto \left(\frac{1}{J(\varphi)}\frac{\det O}{e(\sigma_0)\cdot O \cdot e(\sigma_1)}\right)^{1/2}_{\tau=\tau_{s.p.}} \exp\left[ik_0 \int_0^S n_{refr} ds\right] \Psi(s,\zeta) \tag{153}$$

where

$$\left[2ik_0 n_{refr}\frac{\partial}{\partial s} + \rho\frac{\partial}{\partial\zeta}\left(\frac{1}{\rho}\frac{\partial}{\partial\zeta}\right) - 2k_0^2\tilde{V}(s,\zeta)\right]\Psi(s,\zeta) = 0 \tag{154}$$

This parabolic equation differs from Eq.(134) in two respects–the s-dependent scale factor n_{refr} and the replacement of V with $\tilde{V}$.

6.2 Modal Analysis

Much of the work done to model global propagation has assumed a description involving discrete, local, normal modes propagating without coupling (see, e.g., Munk, Worcester, and Wunsch, 1995). In this section we discuss how this description can be incorporated into the formalism presented in the previous sections.

The local modes satisfy the equation

$$\left[\rho(\zeta)\frac{\partial}{\partial\zeta}\left(\frac{1}{\rho(\zeta)}\frac{\partial}{\partial\zeta}\right) + k_0^2 n^2(\varphi,\lambda,\zeta)\right] Z_m(\varphi,\lambda,\zeta) = k_0^2 \varkappa_m^2(\varphi,\lambda) Z_m(\varphi,\lambda,\zeta) \tag{155}$$

where $\varkappa_m$ is the (dimensionless) normal-mode wave number. These modes are assumed to be orthogonal

$$\int \frac{d\zeta}{\rho(\zeta)} Z_n(\varphi,\lambda,\zeta) Z_m(\varphi,\lambda,\zeta) = \delta_{nm}$$

and complete

$$\sum_m Z_m(\varphi,\lambda,\zeta) Z_m(\varphi,\lambda,\zeta_0) = \rho(\zeta_0)\delta(\zeta-\zeta_0)$$

We expand Φ in Eq.(56) in terms of these modes

$$\Phi(\tau;\varphi,\lambda,\zeta) = \frac{1}{\rho(\zeta_0)}\sum_m \mathcal{C}_m(\tau;\varphi,\lambda)Z_m(\varphi,\lambda,\zeta)Z_m(\varphi,\lambda,\zeta_0)$$

so that

$$Lim_{\tau\to 0}\mathcal{C}_m(\tau;\varphi,\lambda) = \frac{1}{J(\varphi)}\delta(\varphi-\varphi_0)\,\delta(\lambda-\lambda_0)$$

independent of the index m. We have for the Green's function

$$G(\vec{x}\,|\,\vec{x}_0) = \frac{1}{\rho(\zeta_0)}\sum_m Z_m(\varphi,\lambda,\zeta)Z_m(\varphi,\lambda,\zeta_0)G_m(\varphi,\lambda\,|\,\varphi_0,\lambda_0) \tag{156}$$

where

$$G_m = \frac{i}{2k_0}\int_0^\infty d\tau\, e^{i\tau k_0/2}\mathcal{C}_m(\tau;\varphi,\lambda) \tag{157}$$

The equation satisfied by the model coefficients is

$$\left[2ik_0\frac{\partial\mathcal{C}_m}{\partial\tau} + \nabla_S^2\mathcal{C}_m - 2k_0^2W_m\mathcal{C}_m\right]Z_m(\varphi,\lambda,\zeta_0) + \sum_l\left[B_{ml}\mathcal{C}_l + 2\vec{F}_{ml}\cdot\nabla_S\mathcal{C}_l\right] = 0 \tag{158}$$

where

$$W_m(\varphi,\lambda) = -\frac{1}{2}\left[\varkappa_m^2(\varphi,\lambda) - 1\right]$$

$$B_{ml} = \int\frac{d\zeta}{\rho}Z_m\nabla_S^2\left(Z_l(\varphi,\lambda,\zeta)Z_l(\varphi,\lambda,\zeta_0)\right)$$

and

$$\vec{F}_{ml} = \int\frac{d\zeta}{\rho}Z_m\nabla_S\left(Z_l(\varphi,\lambda,\zeta)Z_l(\varphi,\lambda,\zeta_0)\right)$$

Generally speaking, a modal analysis is useful only if mode coupling can be ignored. If this is the case, then $B_{ml} = 0$ and $\vec{F}_{ml} = 0$ giving

$$\left[2ik_0\frac{\partial}{\partial\tau} + \nabla_S^2 - 2k_0^2W_m\right]\mathcal{C}_m = 0 \tag{159}$$

We want to develop a method for solving this equation based on the ideas introduced earlier. There are three obvious possibilities. First one could directly solve Eq.(159) using the technicques in Section 6.1 since it has the same form as Eq.(137). Each function G_m then would have the form of Eq.(153) with Ψ set equal to unity and n_{refr} replaced with $\varkappa_m$. Each normal mode would be associated with a different horizontal ray path. This approach corresponds to the ray-mode approach introduced by Burridge and Weinberg (1977).

The second possibility is to write

$$\mathcal{C}_m(\tau;\varphi,\lambda) = \Psi_m(\tau;\varphi,\lambda)H(\tau;\varphi,\lambda) \tag{160}$$

where H obeys

$$\left[2ik_0\frac{\partial}{\partial\tau}+\nabla_S^2\right]H=0$$

and

$$Lim_{\tau\to 0}H(\tau;\varphi,\lambda)=\frac{1}{J(\varphi)}\delta\left(\varphi-\varphi_0\right)\delta\left(\lambda-\lambda_0\right)$$

The ray acoustics solution for H is given by Eq.(126). The equation satisfied by Ψ_m is

$$2ik_0\frac{\partial\Psi_m}{\partial\tau}+\nabla_S^2\Psi_m+2\frac{\nabla_S H}{H}\cdot\nabla_S\Psi_m-2k_0^2W_m\Psi_m=0 \tag{161}$$

and

$$Lim_{\tau\to 0}\Psi_m(\tau;\varphi,\lambda)=1$$

We know

$$\frac{2}{H}\nabla_S H=\frac{2ik_0S}{\tau}\left[\mu\frac{dq_\varphi}{ds}(\sigma_1)\hat{\varphi}\ +\nu\cos\varphi\frac{dq_\lambda}{ds}(\sigma_1)\hat{\lambda}\right]=\frac{2ik_0S}{\tau}\hat{s}(\sigma_1) \tag{162}$$

and

$$\frac{S}{\tau}\frac{\partial}{\partial s}=\frac{\partial}{\partial\tau}+\frac{S}{\tau}\hat{s}(\sigma_1)\cdot\nabla_\perp \tag{163}$$

so that

$$2ik_0\frac{S}{\tau}\frac{\partial\Psi_m}{\partial s}+\nabla_S^2\Psi_m-2k_0^2W_m\Psi_m=0 \tag{164}$$

The horizontal Laplacian operator could be resolved into components tangent to the path and normal to it and then the second-derivative tangent term would be discarded while the second-derivative normal term is retained. This would give a two-dimensional parabolic equation to solve with a first derivative term along the path and a second derivative term normal to it. This is similar to the approach followed in several studies in global propagation (Collins, 1993b; McDonald *et al.*, 1994; Collins *et al.*, 1995; 1996). The difficulty is that we have already made the horizontal ray acoustics approximation in determing H. To not make the same approximation in the equation for Ψ_m is somewhat inconsistent and results in an approximate equation that has the same domain of validity as if we had consistently made the approximation. The smooth variation of the medium in the horizontal dictates the path taken by the wave. In this approach the path of the wave dictates in what direction the medium can be treated as being smoothly varying.

The third possibility is to consistently apply the horizontal ray approximation and replace Eq.(164) with

$$2ik_0\frac{S}{\tau}\frac{\partial\Psi_m}{\partial s}-2k_0^2W_m\Psi_m=0 \tag{165}$$

giving

$$\Psi_m=\exp\left[-ik_0\frac{\tau}{S}\int_0^S dsW_m)\right] \tag{166}$$

Combining terms

$$G_m = \frac{1}{4\pi}\left(\frac{R_{ave}}{R(S)}\frac{R_{ave}}{R(0)}\right)^{1/4}\left[\frac{S/R_{ave}}{\sin(S/R_{ave})}\right]^{1/2}\int_0^\infty \frac{d\tau}{\tau}\exp\left[\frac{ik_0}{2}\left(\frac{S^2}{\tau}+\tau\left\langle \varkappa_m^2\right\rangle\right)\right]$$

where $\langle \varkappa_m^2 \rangle$ is the mean-square-average of $\varkappa_m$ over the horizontal ray path

$$\left\langle \varkappa_m^2\right\rangle = \frac{1}{S}\int_0^S ds \varkappa_m^2 \tag{167}$$

For $k_0 S >> 1$ the integral over τ is approximately equal to

$$\left(\frac{2\pi i}{k_0 S^2}\right)^{1/2}\left(\frac{\langle \varkappa_m^2\rangle}{S^2}\right)^{1/4}\exp\left[ik_0 S\sqrt{\langle \varkappa_m^2\rangle}\right]$$

so that

$$G_m = \frac{1}{4\pi}\left(\frac{R_{ave}}{R(S)}\frac{R_{ave}}{R(0)}\frac{\langle \varkappa_m^2\rangle}{S^2}\right)^{1/4}\left[\frac{2\pi i}{k_0 S R_{ave}\sin(S/R_{ave})}\right]^{1/2}\exp\left[ik_0 S\sqrt{\langle \varkappa_m^2\rangle}\right] \tag{168}$$

7. SUMMARY

In this report we have proposed a new method for obtaining parabolic approximations. It consists of the following steps:

1. The Green's function for the problem of interest is written as a Laplace or Mellon transform over the corresponding propagator. The transform variable is the Fock-Feynman parameter.

2. The propagator is factored into a term that is assumed to be a rapidly varying function of the horizontal coordinates and obeys a parabolic equation in those coordinates, and an envelope function that is assumed to be a slowly varying function of the horizontal coordinates.

3. The dependence on the horizontal coordinates is determined using the ray acoustics approximation.

4. The equation for the envelop function is cast into the form of a parabolic equation with the horizontal direction of propagation along the ray path determined in step 3.

5. The integral over the Fock-Feynman parameter is approximated using the method of stationary phase.

If one applies this method to the Helmhotz equation in Cartesian coordinates, one obtains the standard narrow-angle parabolic equation in two coordinate variables. One can easily derive improved approximations because the proposed method separates the parabolic approximation into two separate approximations: one involving the characteristics of the acoustic field in the horizontal and the other its characteristics in the vertical. The horizontal ray-acoustics approximation can be relaxed in any number of ways. One possibility is to use the Rytov approximation instead of the ray-acoustics one (Palmer, 1976). This results in replacing the index of refraction by an effective one that is constructed by integration over the Fresnel ray tube surrounding the ray path. Another possibility is to allow for horizontal refraction and horizontal multipaths. In the previous section we indicate how this might be done.

The narrow-angle approximation is easy to relax because there are many ways of improving on the stationary-phase approximation. One possibility is to evalute the integral numerically in the same way that the integral over frequency is evaluated in the Fourier representation of the solution to the parabolic equation for a broadband source. One obtains the solution to Eq.(92) for a range of values of τ using a marching algorithm. The integral in Eq.(92) is then evaluated using a fast Fourier transform. Other than those relating to the numerical evaluation, the only approximations made in the development of the global problem considered here are the horizontal ray acoustics approximation and, of less importance, the WKB approximation for the amplitude.

8. REFERENCES

Babič, V. M., and V. S. Buldyrev, 1991, Short-Wavelength Diffraction Theory, Asymptotic Methods, (Springer-Verlag, New York, 1991) Chap. 6.

Böhm, M., and Georg Junker, "Path integration over compact and noncompact rotation groups," J. Math Phys. **28**, 1978-1994 (1987).

Brekhovskikh, L. M., and O. A. Godin, 1999, Acoustics of Layered Media II, Point Sources and Bounded Beams (Springer-Verlag, New York, 1999).

Bremmer, H., 1951, "The W.K.B. approximation as the first term of a geometric-optics series," Comm. Pur and Appl. Math. **4**, 105-115 (1951).

Burridge, R., and H. Weinberg, 1977, "Horizontal rays and vertical modes," in Wave Propagation and Underwater Acoustics, Lecture Notes in Physics, Vol. 70, edited by J. B. Keller and J. S. Papadakis (Springer-Verlag, New York, 1977)

Collins, M. D., 1991, "Higher-order parabolic approximations for accurate and stable elastic parabolic equations with application to interface wave propagation," J. Acoust. Soc. Am. **90**, 1050-1057 (1991).

Collins, M. D., 1993a, "An energy-conserving parabolic equation for elastic media," J.Acoust. Soc. Am. **94**, 975-982 (1993).

Collins, M. D., 1993b, "The adiabatic mode parabolic equation," J. Acoust.Soc. Am. **94**, 2269-2278 (1993).

Collins, M. D., B. E. McDonald, K. D. Heaney, and W. A. Kuperman, 1995a, "Three-dimensional effects in global acoustics," J. Acoust. Soc. Am. **97**, 1567-1575 (1995).

Collins, M. D., B. E. McDonald, W. A. Kuperman, and W. L. Siegmann, 1995b, "Jovian acoustics and Comet Shoemaker-Levy 9," J. Acoust. Soc. Am. **97**, 2147-2158 (1995).

Collins, M. D., B. E. McDonald, W. A. Kuperman, and W. L. Siegmann, 1996, "Erratum: Jovian acoustics and Comet Shoemaker-Levy 9," J. Acoust. Soc. Am. **100**, 3983-3984 (1996).

Corones, J., 1982, "Bremmer Series that Correct Parabolic Approximations," J. Math Anal. Appl. **50**, 361-372 (1982).

Corones, J., B. DeFacio and R. J. Krueger, 1982, "Parabolic approximations to the time-independent elastic wave equation," J. Math Phys. **23**, 577-586 (1982).

Dworski J., and J. A. Mercer, 1990, "Hamiltonian 3-D Ray Tracing in the Oceanic Waveguide on the Ellipsoidal Earth," Applied Physics Laboratory, University of Washington Technical Report APL-UW TR 8929 (1990).

Feynman, R. P., 1951, "An operator calculus having application in quantum electrodynamics," Phys. Rev. **84**, 108 (1951).

Fock, V., 1937, Phys. Z. Sowjetunion **12**, 404 (1937).

Greene, R. R., 1985, "A high-angle one-way wave equation for seismic wave propagation along rough and sloping interfaces," J. Acoust. Soc. Am. **87**, 1991-1998 (1985).

Godin, O. A., 1987, "A new form of the wave equation for sound in a general layered fluid," in Progress in Underwater Acoustics, edited by H. M. Merklinger (Plenum Press, New York, 1987) pp. 337-349.

Godin, O. A., 1991, "Parabolic approximationd in acoustics of moving media," Akust. Zh. **37**, 646-653 (1991) [English translation in Sov. Phys. Acoust. **37**, 335-338 (1991)].

Godin, O. A., and A. V. Mokhov, 1992, "Small-angle parabolic approximation for sound fields in a moving fluid," Akust. Zh. **38**, 442-449 (1992) [English translation in Sov.Phys. Acoust. **38**, 243-246 (1992)].

Godin, O. A., 1998a, "Wave equations for unidirectional propagation of sound in a moving fluid," Doklady Physics **43**, 461-464 (1998).

Godin, O. A., 1998b, "Energy-conserving and reciprocal parabolic approximation," in .Proceedings of the Fourth European Conference on Underwater Acoustics, Rome, Italy, September 1998, Volume 2. (CNR_IDAC, Rome, 1998), pp 679-684.

Godin, O. A., 1999, "A wide-angle, energy-conserving parabolic equation for sound in moving medium," in Proceedings of the Third International Conference on Theoretical and Computational Acoustics (World Scientific, New Jersey, 1999).

Gutzwiller, M. C., 1990, Chaos in Classical and Quantum Mechanics (Springer-Verlag, New York, 1990), Eq.(2.10).

Hardin, R. H., and F. D. Tappert, 1973, "Applications of the Split-Step Fourier Method to the Numerical Solution of Nonlinear and Variable Coefficient Wave Equations," SIAM Rev. **15**, 423 (1973).

Hudson, J. A., 1980, "A parabolic approximation for elastic waves," Wave Motion **2**, 207-214 (1980).

Kreigsmann, G. A., 1985, "A multiscale derivation of a new parabolic equation which includes density variations," Comp. Math Appl. **11**, 817-821 (1985).

Landers, T., and J. F. Claerbout, 1972, "Numerical calculations of elastic waves in laterally inhomogeneous media," J. Geophys. Res. **77**, 1476-1482 (1972).

McAninch, G. L., 1986, "Higher order parabolic approximations of the reduced wave equation," J. Sound and Vibration **104**, 411-424 (1986).

McDonald, B. E., M. D. Collins, W. A. Kuperman, and K. D. Heaney 1994, "Comparison of data and model predictions for Heard Island acoustic transmissions," J. Acoust. Soc. Am. **96**, 2357-2370 (1994).

Munk, W., P. Worcester, and C. Wunsch, Ocean Acoustic Tomography (Cambridge University Press, New York, 1995).

Myers, M. K., and G. L. McAninch, 1978, "Parabolic Approximation for Sound Propagation in the Atmosphere," AIAA Journal **16**, 836-842 (1978).

Orchard, B. J., W. L. Siegmann, and M. J. Jacobson, 1992, "Three-dimensional time domain paraxial approximations for ocean acoustic wave propagation," J. Acoust. Soc. Am. **91**, 788-801 (1992).

Palmer, D. R., 1976, "Eikonal approximation and the parabolic equation," J. Acoust.

Soc. Am. **60**, 343-354 (1976).

Palmer, D. R., 1978, "An Introduction to the Application of Feynman Path Integrals to Sound Propagation in the Ocean," Nav. Res. Lab. Rep. 8148 (1978) (NTIS Document ADA 060734, National Technical Information Service, Springfield, VA).

Palmer, D. R., 1979, "A path-integral approach to the parabolic equation. I," J. Acoust. Soc. Am. **66**, 862-871 (1979).

Siegmann, W. L., G. A. Kreigsmann, and D. Lee, 1985, "A wide-angle three-dimensional parabolic wave equation," J. Acoust. Soc. Am. **78**, 659-664 (1985).

Tappert, F. D., and R. H. Hardin, in "A Synopsis of the AESD Workshop on Acoustic Modeling by Non-Ray Tracing Techniques," AESD Tech. Note TN 73-05 (Nov. 1973).

Tappert, F. D., "The Parabolic Approximation Method," in Wave Propagation and Underwater Acoustics, Lecture Notes in Physics, Vol. 70, edited by J. B. Keller and J. S. Papadakis (Springer-Verlag, New York, 1977).

Tappert, F., J. L. Spiesberger, and L. Boden, 1995, "New full-wave approximations for ocean acoustic travel time predictions," J. Acoust. Soc. Am. **97**, 2771-2782 (1995).

Wales, S. C., and J. J. McCoy, 1983, "A comparison of parabolic wave theories for linearly elastic solids," Wave Motion **5**, 99-113 (1983).

Weinberg, S., 1962, "Eikonal method in magnetohydrodynamics," Phys. Rev. **126**, 1899-1909 (1962).

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APPENDIX LIST OF SYMBOLS

$A(\tau;\varphi,\lambda)$	Eikonal function
$\tilde{A} = A - \tau\partial A/\partial\tau$	Horizontal eikonal function
$A_\varphi = p_\varphi h_\lambda/h_\varphi$	Normalized conjugate momenta
$A_\lambda = p_\lambda h_\varphi/h_\lambda$	Normalized conjugate momenta
B_{ml}	Scalar mode coupling function
$c(\vec{x})$	Speed of sound at the point $\vec{x}$
c_0	Reference sound speed
$\mathcal{C}$	Integration constant
$\mathcal{C}_m(\tau;\varphi,\lambda)$	Coefficient in the modal expansion of Φ
$D = J(q_\varphi)\det(Q)$	
$D_2 = (\sigma-\sigma_0)W^2/A_\varphi$	
$D_1 = D/(\sigma-\sigma_0)$	
D_1^{WKB}	WKB approximation to D_1
$d\vec{x}_S$	Differential line segment tangent to the surface
$ds = \lvert d\vec{x}_S \rvert$	Differential arc length
$e = \begin{bmatrix} dq_\varphi/d\sigma \\ dq_\lambda/d\sigma \end{bmatrix}$	Two-dimensional column vector
$\vec{F}_{ml}$	Vector mode-coupling function
$G(\vec{x}\,\vert\,\vec{x}_0)$	Green's function
$G_m(\varphi,\lambda\,\vert\,\varphi_0,\lambda_0)$	Coefficient in the modal expansion of the Green's function
$g(\varphi)$	Deviation of the Earth's shape from that of a sphere
$H(\tau;\varphi,\lambda)$	Solution to the horizontal parabolic equation
$H_\varphi = (\nu\cos\varphi/\mu)\,\partial H/\partial\varphi$	
$H_\lambda = (\mu/\nu\cos\varphi)\,\partial H/\partial\lambda$	
$\left(\hat{H}\ \hat{H}_\varphi\ \hat{H}_\lambda\right) = (H\ H_\varphi\ H_\lambda)\,e^{-ik_0A}$	Slowly varying amplitude functions
$h_\varphi = \mu(q_\varphi)$	Introduced to simplify the notation
$h_\lambda = \nu(q_\varphi)\cos(q_\varphi)$	Introduced to simplify the notation

$J(\varphi) = \mu(\varphi)\nu(\varphi)\cos(\varphi)$	Jacobian
k_0	Reference wave number
$n(\vec{x}) = c_0/c(\vec{x})$	Index of refraction
n_{refr}	Depth-independent reference index of refraction
$\hat{n} = \hat{\lambda} \times \hat{\varphi}_g$	Unit vector normal to the surface
O	2×2 matrix consisting of second derivatives of the eikonal
$p(\vec{x})$	Complex demodulated pressure field
$P = p_\lambda / W$	Quantity constant along the ray path
$p_\tau,\ p_\varphi,\ p_\lambda$	Conjugate momenta to the ray paths
$q_\tau,\ q_\varphi,\ q_\lambda$	Ray path coordinates
Q	2×2 matrix introduced to help solve the transport equation
$R(\varphi) = \sqrt{\mu(\varphi)\nu(\varphi)}$	
R_{ave}	$R(q_\varphi)$ averaged over the ray path
$\bar{R}$	Mean radius of the Earth
$s(\sigma)$	Horizontal path length
$S = s(\sigma_1)$	Total horizontal path length
$\hat{s}(\sigma)$	Unit vector in the direction of $d\vec{x}_S$
$U = -\left(n_{refr}^2 - 1\right)/2$	Variation in the reference index of refraction
$V = -\left(n^2 - 1\right)/2$	Variation in the index of refraction
$V(s,\zeta) = V(\varphi_{ray}(s), \lambda_{ray}(s), \zeta)$	Introduced to simplify the notation
$\tilde{V} = V - U$	
$\mathbf{v} = ds/d\sigma$	Rate of change of path length
$\vec{x} = (x, y, z) = (r, \varphi, z)$	Position vector in Cartesian and cylindricalcoordinatesof a general point in the medium
$\hat{x},\ \hat{y},\ \hat{z}$	Unit vectors in a Cartesian system with origin at the center of the Earth, z-axis toward the north pole, and x-axis at zero longitude

$\vec{x}_S = (x_S, y_S, z_S)$	Position vector from the origin of a Cartesian system centered in Earth to a point on its surface.
$\hat{x}_S$	Unit vector in the direction of increasing $\vec{x}_S$
$\vec{x}_0 = (x_0, y_0, z_0)$	Position vector of the a point source
$W \equiv \sqrt{-2\, p_\tau}$	Useful variable constant along the ray path
$W_m(\varphi, \lambda) = -\frac{1}{2}\left[\varkappa_m^2(\varphi, \lambda) - 1\right]$	
$Z_m(\varphi, \lambda, \zeta)$	Local normal-mode eigenfuction
α	Angle between $d\vec{x}_S$ and north
∇	Gradient operator
∇^2	Laplacian operator
∇_S^2	Horizontal Laplacian operator
$\delta^{(3)}\left(\vec{x} - \vec{x}_0\right)$	3-D Dirac delta function
$\rho_S = \sqrt{x_S^2 + y_S^2}$	
$\rho = \rho(\zeta)$	Density of the medium
$\sigma\ ,\ (\sigma_0 \leq \sigma \leq \sigma_1)$	Marks evalution along the ray path
τ	Fock-Feynman parameter
$\tau_{s.p.}$	Stationary-phase point
$\tau_{ray}(s), \varphi_{ray}(s), \lambda_{ray}(s)$	Ray path coordinates as a function of path length
$\Phi(\tau; \varphi, \lambda, \zeta) = H(\tau; \varphi, \lambda)\Psi(\tau; \varphi, \lambda, \zeta)$	Propagator function
φ	Geodetic latitude
φ_g	Geocentric latitude
ζ	Depth variable
η	Deviation of the Earth's shape from a sphere, $\eta \approx 1/150$
$\varkappa_m(\varphi, \lambda)$	Local, dimensionless, normal-mode wave number

$\langle \varkappa_m^2 \rangle$	Mean-square-average of $\varkappa_m$ over the horizontal ray path
λ	Longitude
$\hat{\lambda}$, $\hat{\rho}$, $\hat{\varphi}_g$, $\hat{\varphi}$	Unit vectors on the Earth's surface in the directions of increasing $\lambda, \rho_S, \varphi_g$,and φ, respectively
μ	Meridional radius of curvature
$\mu_0 = \mu(\varphi_0)$	Values of μ at the source location
ν	Prime vertical radius of curvature
$\nu_0 = \nu(\varphi_0)$	Value of ν at the source location
$\xi = \sqrt{(d\rho_S/d\varphi_g)^2 + (dz_S/d\varphi_g)^2}$	
$\Psi_\varphi = (\nu \cos\varphi/\mu)\, \partial\Psi/\partial\varphi$	
$\Psi_\lambda = (\mu/\nu \cos\varphi)\, \partial\Psi/\partial\lambda$	
χ	Function defined by $z_S = \chi \sin\varphi$
$\Psi(\tau; \varphi, \lambda, \zeta)$	Envelope function satisfying the curvilinear parabolic equation
$\Psi_\zeta = (1/\rho)\, \partial\Psi/\partial\zeta$	
ψ	Envelope function satisfying the 2-D parabolic equation
$\psi(s, \zeta) = \Psi(S; \varphi_{ray}(s), \lambda_{ray}(s), \zeta)$	Introduced to simplify the notation
$\Psi_m(\tau; \varphi, \lambda)$	Modal envelope function
$\Omega = J^{-1}\partial A_\varphi/\partial q_\varphi$	
ω	Angular frequency of the source

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