**Some notes on Montgomery potential in layer ocean models**

Sang-Ki Lee

CIMAS, University of Miami, Miami, FL

1. Derivation of Montgomery potential for isopycnic layer model

The Montgomery potential is the potential energy term of the Bernoulli function. The fluid particle tends to follow the constant Montgomery potential when the kinetic energy term in the Bernoulli function is of much less importance. The Montgomery potential can be also recognized as the pressure deficit caused by the fact that the water column above the layer *k* has a density that is different from the one of this layer. The Montgomery potential is defined as

$$\rho M = p + \rho g z \tag{1.1}$$

Consider a hydrostatic ocean, which is divided into N layers of different densities. Neglecting the atmospheric pressure and using Boussinesq approximation, more practical formula for the Montgomery potential is

$$\rho_r M_k = \rho_r g \xi_1 + g \sum_{i=1}^{k-1} (\rho_{i+1} - \rho_i) \xi_{i+1}, \tag{1.2}$$

where $\xi_1$ represents sea surface elevation, and $\xi_k$ the layer interface elevation. The constant terms are removed from (1.2) because they do not contribute on the horizontal gradient. Equation (1.2) can be rearranged to the following recursive equations.

$$\rho_r M_{k+1} = \rho_r M_k + g(\rho_{k+1} - \rho_k)\xi_{k+1} \tag{1.3}$$

$$\rho_r M_{k+1} = \rho_r M_k + g(\rho_{k+1} - \rho_k) \sum_{i=k+1}^{N} h_i \tag{1.4}$$

$$\rho_r M_{k+1} = \rho_r M_k - g(\rho_{k+1} - \rho_k) \left( \sum_{i=1}^{k} h_i - \xi_1 \right) \tag{1.5}$$

Since the isopycnal layer models use pressure rather then interface elevation or thickness variation, equation (1.5) can be written in term of pressure variable.

$$\rho_r M_{k+1} = \rho_r M_k - \alpha_r (\rho_{k+1} - \rho_k)(p_k - \rho_r g \xi_1) \tag{1.6}$$

where the pressure variables are defined at the layer interfaces, i.e.,

$$p_k = \sum_{i=1}^{k} \Delta p_k = \rho_r g \sum_{i=1}^{k} h_k \tag{1.7}$$

Note that, in (1.6) the last term (surface elevation term) can be neglected since it is multiplied with ($\rho_{k+1}$ - $\rho_k$). Then the Montgomery potential can be now written as

$$\rho_r M_{k+1} = \rho_r M_k - g(\rho_{k+1} - \rho_k) p_k \tag{1.8}$$

Suppose that the surface elevation is not known, but the bottom pressure is known. Then the Montgomery potential at the bottom is, by definition,

$$\rho_r M_N = p_b - \rho_r g H \tag{1.9}$$

where the bottom pressure is defined as

$$p_b = \sum_{k=1}^{N} \Delta p_k \ . \tag{1.10}$$

Now, we can derive the Montgomery potential, starting from the bottom, using (1.2) and (1.9).

$$
\begin{aligned}
\rho_r M_k - \rho_r M_N &= \rho_r g \xi_1 + g \sum_{i=1}^{k-1} (\rho_{i+1} - \rho_i) \xi_{i+1} - \rho_r g \xi_1 - g \sum_{i=1}^{N-1} (\rho_{i+1} - \rho_i) \xi_{i+1} \\
&= -g \sum_{i=k}^{N-1} (\rho_{i+1} - \rho_i) \xi_{i+1}
\end{aligned} \tag{1.11}
$$

Using the known Montgomery potential at the bottom (1.9), we get

$$
\begin{aligned}
\rho_k M_k &= p_b - \rho_r g H - g \sum_{i=k}^{N-1} (\rho_{i+1} - \rho_i) \xi_{i+1} \\
&= p_b - \rho_r g H - g \sum_{i=k}^{N-1} \left[ (\rho_{i+1} - \rho_i) \sum_{j=i+1}^{N} h_j \right] \\
&= p_b - \rho_r g H + g \sum_{i=k}^{N-1} \left[ (\rho_{i+1} - \rho_i) \left( \sum_{j=1}^{i+1} h_j - \xi_1 \right) \right] \\
&= p_b - \rho_r g H + \frac{1}{\rho_r} \sum_{i=k}^{N-1} (\rho_{i+1} - \rho_i)(p_{i+1} - \rho_r g \xi_1)
\end{aligned} \tag{1.12}
$$

Neglecting the small term in the last expression of (1.12), we get the final form of the Montgomery potential to be used in the model.

$$
\rho_r M_k = p_b - \rho_r g H + \frac{1}{\rho_r} \sum_{i=k}^{N-1} (\rho_{i+1} - \rho_i) p_{i+1} \tag{1.13}
$$

2. Mode splitting of Hallberg (1997)

In Hallberg (1997), the barotropic Montgomery potential was obtained by vertically integrating (1.13) from top to the bottom of the ocean. However, we are interested in the vertical average of the horizontal gradient, which can be written as

$$
\rho_r \frac{\partial \overline{M}}{\partial x} = \frac{\partial p_b}{\partial x} + \frac{1}{p_b} \sum_{k=1}^{N} \left[ \Delta p_k \frac{1}{\rho_r} \sum_{i=k}^{N-1} (\rho_{i+1} - \rho_i) \frac{\partial p_{i+1}}{\partial x} \right] \tag{2.1}
$$

Note that the contribution of surface elevation in the pressure variables in the last term is very small, due to the multiplicative factor ($\rho_{k+1}$ - $\rho_k$). Therefore it is neglected.

3. Mode splitting of Bleck and Smith (1990).

In Bleck and Smith (1990), it was assumed that the divergence caused by external mode in each layer is proportional to that of the external mode. Therefore, the pressure was expressed in the following way.

$$p = p'(1+\eta) \tag{3.1}$$

where $\eta$ is dimensionless, and it represents the external mode component of the pressure field. Substituting this equation into (1.9) gives,

$$\rho_r M_N = p_b'(1+\eta) - \rho_r gH \tag{3.2}$$

where $p_b'$ is non-time varying and can be written as

$$p_b' = \sum_{k=1}^{N} \Delta p_k - \rho_r g \xi_1 \tag{3.3}$$

The baroclinic Montgomery potential at the bottom must be also non-time varying, and it is defined as

$$\rho_r M'_N = \rho_r M_N - p_b'\eta = p_b' - \rho_r gH \tag{3.4}$$

In any case, substituting (3.1) into (1-13), the Montgomery potential for each layer becomes

$$\rho_r M_k = p_b'\eta + \frac{1}{\rho_r}\sum_{i=k}^{N-1}(\rho_{i+1} - \rho_i)p'_{i+1}\,. \tag{3.5}$$

Note that $\eta(\rho_{k+1} - \rho_k)$ terms are neglected since they are small.

The baroclinic Montgomery potential is therefore defined as

$$\rho_r M'_k = \frac{1}{\rho_r}\sum_{i=k}^{N-1}(\rho_{i+1}-\rho_i)p'_{i+1} \tag{3.6}$$

4. Major differences between Hallberg (1997) and Bleck and Smith (1990)

It is obvious that the barotropic Montgomery potential used in Hallberg (1997) and Bleck and Smith (1990) are very much different. Hallberg (1997) defined the barotropic Montgomery potential as “depth averaged”, which is consistent with the definition of barotropic transport. In Bleck and Smith (1990), on the other hand, the barotropic Montgomery potential is defined as “the bottom pressure”. Due to the inconsistency between the definition of “barotropic Montgomery potential” and “barotropic transport”, Bleck and Smith (1990) had to introduce fictitious terms, $u^*$ and $v^*$ in the momentum equations in order to counterbalance this inconsistency.

It is apparent that the depth average of the baroclinic Montgomery potential (3.6) is non-zero. Therefore, equations for $u^*$ and $v^*$ can be derived.

$$\rho_r \frac{\partial M''_k}{\partial x} = \frac{\partial p'_b \eta}{\partial x} + \frac{1}{\rho_r}\sum_{i=k}^{N-1}(\rho_{i+1}-\rho_i)\frac{\partial p'_{i+1}}{\partial x} - \frac{\partial u^*}{\partial t}, \tag{4.1}$$

where $M''$ is the baroclinic Montgomery potential defined in Hallberg (1997) in which the depth average is zero. Using (1.13) and (2.1), we get the equation for $u^*$.

$$\frac{\partial u^*}{\partial t} = \frac{\partial p'_b \eta}{\partial x} + \frac{1}{p_b}\sum_{k=1}^{N}\left[\Delta p_k \frac{1}{\rho_r}\sum_{i=k}^{N-1}(\rho_{i+1}-\rho_i)\frac{\partial p_{i+1}}{\partial x}\right] \tag{4.2}$$

Following the same method, $v^*$ can be also obtained.

In MICOM and also in HYCOM, $u^*$ is kept constant while the barotropic Montgomery potential and momentum are integrated in time. Obviously, $u^*$ term contains barotropic part as well as baroclinic part of the Mongomery potential. Since $u^*$ contains fast mode, it has to be also updated during the calculation of fast mode (barotropic mode). This is a flaw in MICOM and HYCOM, and as pointed out by Higdon and Bennett (1996) and also by de Szoeke and Higdon (1997), numerical instability occurs at all wavelengths for any size time step due to the inexact splitting between the fast and slow modes. Hallberg (1997) pointed out that heavy temporal smoothing is required in MICOM and HYCOM in order to remove this numerical instability. Hallberg (1997) also pointed out that mass conservation for each layer may not be achieved due to this flaw for long term time integration.

## 5. Application to two–layers model

We understand that $u^*$ and $v^*$ in MICOM and HYCOM are basically pressure gradient terms and plays very important roles. These terms are diagnostically determined at each time step by the equation stating that the depth averaged baroclnic transport must be zero.

Now, let's see how the model works when the data are assimilated. To understand, two-layer model is a useful too. Let's assume geostropy to further simply the dynamics. For two-layer model, the zonal momentum equations are

$$-fv_1 = -g\frac{\partial \xi_1}{\partial x}, \tag{5.1}$$

$$-fv_2 = -g\frac{\partial \xi_1}{\partial x} - g'\frac{\partial \xi_2}{\partial x}. \tag{5.2}$$

Multiplying $h_1$ and $h_2$ to (14) and (15), respectively, we get the zonal momentum equation for the external mode:

$$-f\overline{v} = -g\frac{\partial \xi_1}{\partial x} - g'\frac{\partial \xi_2}{\partial x}\frac{h_2'}{H}, \tag{5.3}$$

Subtracting (5.3) from (5.1) and (5.2) yields zonal momentum equations for internal mode:

$$-fv_1' = g'\frac{\partial \xi_2}{\partial x}\frac{h_2'}{H}. \tag{5.4}$$

$$-fv_2' = -g'\frac{\partial \xi_2}{\partial x}\frac{h_1'}{H}. \tag{5.5}$$

According to Bleck and Smith (1990), the barotropic Montgomery potential is the bottom pressure, i.e.,

$$\alpha_r\frac{\partial p_b'\eta}{\partial x} = g\frac{\partial \xi_1}{\partial x} + g'\frac{\partial \xi_2}{\partial x} \tag{5.6}$$

Now using (3.6), the baroclinic momentum equations become

$$\begin{aligned} -fv_1' &= -\alpha_r^2(\rho_2 - \rho_1)\frac{\partial p_2'}{\partial x} - \frac{\partial u^*}{\partial t} \\ &= g'\frac{\partial \xi_2}{\partial x} - \frac{\partial u^*}{\partial t} \end{aligned}, \tag{5.7}$$

$$-fv_2' = -\frac{\partial u^*}{\partial t}. \tag{5.8}$$

By the definition that the baroclinic transport has to be zero, we get the following equation for $u^*$., i.e.,

$$v_1'\Delta p_1' + v_2'\Delta p_2' = g'\frac{\partial \xi_2}{\partial x}\Delta p_1' - \frac{\partial u^*}{\partial t}p_3' = 0. \tag{5.9}$$

Or

$$\frac{\partial u^*}{\partial t} = g'\frac{\partial \xi_2}{\partial x}\frac{h_1'}{H}. \tag{5.9}$$

The barotropic momentum equation becomes

$$
\begin{aligned}
-f\bar{v} &= -\alpha_r \frac{\partial p_b' \eta}{\partial x} + \frac{\partial u^*}{\partial t} \\
&= -g\frac{\partial \xi_1}{\partial x} - g'\frac{\partial \xi_2}{\partial x} + \frac{\partial u^*}{\partial t}
\end{aligned}
\tag{5.10}
$$

6. Data assimilation (two-layer model)

Let's suppose that the pressure is corrected through data assimilation. If one does no pretreatment in the momentum equation, the momentum equation for the correction is

$$
\begin{aligned}
-f\left(\bar{v} + \Delta\bar{v}\right) &= -\alpha_r \frac{\partial p_b'\left(\eta + \Delta\eta\right)}{\partial x} + \frac{\partial u^*}{\partial t} \\
&= -g\frac{\partial \xi_1}{\partial x} - g'\frac{\partial\left(\xi_2 + \Delta\xi_2\right)}{\partial x} + \frac{\partial u^*}{\partial t}
\end{aligned},
\tag{6.1}
$$

$$
\begin{aligned}
-f\left(v_1' + \Delta v_1'\right) &= -\frac{\partial\left(M_1' + \Delta M_1'\right)}{\partial x} - \frac{\partial u^*}{\partial t} \\
&= g'\frac{\partial\left(\xi_2 + \Delta\xi_2\right)}{\partial x} - \frac{\partial u^*}{\partial t}
\end{aligned}.
\tag{6.2}
$$

$$
-f\left(v_2' + \Delta v_2'\right) = -\frac{\partial u^*}{\partial t}.
\tag{6.3}
$$

Note that the perturbation of the baroclinic Montgomery potential in the 2nd layer is kept zero. Therefore, the perturbation in the 2nd layer has to be enforced through $u^*$ term. The depth integration of the corrected baroclinic momentum fields has to vanish. Then we get the following equation.

$$
\begin{aligned}
&\left(v_1' + \delta v_1'\right)\left(\Delta p_1' + \delta\Delta p_1'\right) + \left(v_2' + \delta v_2'\right)\left(\Delta p_2' + \delta\Delta p_2'\right) \\
&\qquad = g'\frac{\partial\left(\xi_2 + \delta\xi_2\right)}{\partial x}\left(\Delta p_1' + \delta\Delta p_1'\right) - \frac{\partial u^*}{\partial t} p_3' = 0
\end{aligned}.
\tag{6.4}
$$

$$\frac{\partial u^*}{\partial t} = g' \frac{\partial(\xi_2 + \delta\xi_2)}{\partial x} \frac{(\Delta p_1' + \delta\Delta p_1')}{p_3'} . \tag{6.5}$$

Plugging this equation into (6.1)~(6.3), we get

$$- f(\bar{v} + \delta\bar{v}) = -g \frac{\partial \xi_1}{\partial x} - g' \frac{\partial(\xi_2 + \delta\xi_2)}{\partial x} \frac{(\Delta p_2' + \delta p_2')}{p_3'} , \tag{6.6}$$

$$- f(v_1' + \delta v_1') = g' \frac{\partial(\xi_2 + \delta\xi_2)}{\partial x} \frac{(\Delta p_2' + \delta p_2')}{p_3'} . \tag{6.7}$$

$$- f(v_2' + \delta v_2') = -\frac{\partial(\xi_2 + \delta\xi_2)}{\partial x} \frac{(\Delta p_1' + \delta\Delta p_1')}{p_3'} . \tag{6.8}$$

Now, subtract (5.3), (5.4) and (5.5) from (6.6), (6.7) and (6.8), respectively. The momentum equations for the correction emerges,

$$- f\delta\bar{v} = -g' \frac{\partial(\xi_2 + \delta\xi_2)}{\partial x} \frac{(\Delta p_2' + \delta p_2')}{p_3'} + g' \frac{\partial \xi_2}{\partial x} \frac{\Delta p_2'}{p_3'} , \tag{6.9}$$

$$- f\delta v_1' = g' \frac{\partial(\xi_2 + \delta\xi_2)}{\partial x} \frac{(\Delta p_2' + \delta p_2')}{p_3'} - g' \frac{\partial \xi_2}{\partial x} \frac{\Delta p_2'}{p_3'} . \tag{6.10}$$

$$- f\delta v_2' = -g' \frac{\partial(\xi_2 + \delta\xi_2)}{\partial x} \frac{(\Delta p_1' + \delta\Delta p_1')}{p_3'} + g' \frac{\partial \xi_2}{\partial x} \frac{\Delta p_2'}{p_3'} . \tag{6.11}$$

Since the XBT cast does not contain any information on surface elevation, the surface elevation has to be kept unchanged during the assimilation. This can be achieved by the following formula.

$$\rho_r M_1^b = \rho_r M_1'^b + p_b' \eta^b = \rho_r M_1'^a + p_b' \eta^a \tag{6.12}$$

where superscript *b* and *a* represent “before” and “after” the assimilation. By using (3.6), the new barotropic Montgomery potential can be found.

$$p_b'\eta^a = \rho_r M_1'^b + p_b'\eta^b - \frac{1}{\rho_r}\sum_{i=1}^{N-1}(\rho_{i+1} - \rho_i)p_{i+1}' \qquad (6.13)$$