Some notes on the momentum adjustment after data are assimilated in layer models

Sang-Ki Lee

CIMAS, University of Miami, Miami, FL

1. **A scheme for momentum adjustment after data assimilation**

Before the assimilation, $\overline{u}$, $\overline{v}$, u', v', dp, η are available. Immediately after the assimilation, we have new dp. The objective is to estimate new $\overline{u}$, $\overline{v}$, u', v', η. First of all, let's focus on mid-latitude where the flow is mainly geostrophy. In this case, we can derive the following equations:

$$-f\Delta u' = -\frac{\partial \Delta p'}{\rho \partial x} - \frac{\Delta u^*}{\Delta t_b}, \quad (1)$$

$$f\Delta v' = -\frac{\partial \Delta p'}{\rho \partial y} - \frac{\Delta v^*}{\Delta t_b}, \quad (2)$$

where Δt_b is the time step of the internal model (baroclinic). These equations can be rewritten as

$$u'' = \left(\Delta u' - \frac{\Delta u^*}{f \Delta t_b} \right) = \frac{\partial \Delta p'}{f \rho \partial x}, \quad (3)$$

$$v'' = \left(\Delta v' + \frac{\Delta v^*}{f \Delta t_b} \right) = -\frac{\partial \Delta p'}{f \rho \partial y}. \quad (4)$$

Now, by definition of the internal mode, the following property must be satisfied:

$$\sum_{k=1}^{N} \Delta u'_k dp'_k = \sum_{k=1}^{N} \Delta v'_k dp'_k = 0. \quad (5)$$

Therefore, we can derive the following equations:

$$\Delta u^* = -\frac{f \Delta t_b}{p_b} \sum_{k=1}^{N} u'' dp, \quad (6)$$

$$\Delta v^* = \frac{f \Delta t_b}{p_b} \sum_{k=1}^{N} v'' dp. \quad (7)$$

It is important to point out that Δu^*, Δv^* must vanish if geostrophy holds as in (1) and (2). In any case, now, we will plug new u', v', Δu^*, Δv^* into external momentum equations:

$$-f\bar{v} = -\frac{\partial p_b' \eta}{\rho \partial x} + \frac{u^*}{\Delta t_b}, \tag{8}$$

$$f\bar{u} = -\frac{\partial p_b' \eta}{\rho \partial y} + \frac{v^*}{\Delta t_b}. \tag{9}$$

Then, we can again plug new $\bar{u}$, $\bar{v}$ into the continuity equation for the external mode to get updated η:

$$\frac{\partial p_b' \eta}{\partial t} = -\left(\frac{\partial \bar{u} p_b'}{\partial x} + \frac{\partial \bar{v} p_b'}{\partial y} \right). \tag{10}$$

It is important to note that by removing time dependent terms in (8) and (9), external gravity waves are filtered out, therefore, (10) along with (8) and (9) forms an elliptic equation for $p'_b \eta$.

2. An ad hoc scheme for momentum adjustment near equator

Since geostrophy breaks down near the equator, the momentum fields are not modified. The flowing scheme is used:

$$u' = u_o' + \Delta u' \cdot \left[1 - \exp\left(-\frac{y^2}{R^2} \right) \right], \tag{11}$$

$$v' = v_o' + \Delta v' \cdot \left[1 - \exp\left(-\frac{y^2}{R^2} \right) \right]. \tag{12}$$

Here, since integration of the u_o', v_o' and $\Delta u'$, $\Delta v'$ over new dp' vanish, integration of the new momentum fields also vanish (no net transport).

3. Additional issues

However, there is couple of issues to point out. First of all, we have to make sure that the following properties are satisfied before the assimilation:

$$\sum_{k=1}^{N} u'_k \, dp'_k = \sum_{k=1}^{N} v'_k \, dp'_k = 0 . \tag{13}$$

Otherwise, mass continuity in each layer cannot be conserved. Next point is that the old (before the assimilation) internal momentum fields must be modified even before the momentum correction to satisfy (13) under the corrected dp' fields. This can be achieved by redistributing momentum across the adjacent layers. Of course, a question remains if this procedure is really necessary. But, if this procedure is skipped, it is required to estimate u^* and v^* from old u' and v' and new dp', then add them to the external momentum equations (8) and (9). It is not clear which way is less shock to the numerical model. The last point is the mass continuity issue. This is automatically satisfied if the momentum correction $\Delta u'$, $\Delta v'$ are in geostrophic balance as in (1) and (2).

An alternative method to adjust momentum is to simply estimate new u' and v' by iteration, calculate new u^* and v^* using usual method, then relax the external mode variables ($\bar{u}$, $\bar{v}$, η) in each iteration.

4. Properties of Internal and External modes

For two-layer model, the zonal momentum equations are

$$-fv_1 = -g\frac{\partial \eta_1}{\partial x} , \tag{14}$$

$$-fv_1 = -g\frac{\partial \eta_1}{\partial x} - g'\frac{\partial \eta_2}{\partial x} . \tag{15}$$

Multiplying h_1 and h_2 to (14) and (15), respectively, we get the zonal momentum equation for the external mode:

$$- f\bar{v} = -g\frac{\partial \eta_1}{\partial x} - g'\frac{\partial \eta_2}{\partial x}\frac{h_2'}{H}, \tag{16}$$

Subtracting (16) from (14) and (15) yields zonal momentum equations for internal mode:

$$- fv_1' = g'\frac{\partial \eta_2}{\partial y}\frac{h_2'}{H}. \tag{17}$$

$$- fv_2' = -g'\frac{\partial \eta_2}{\partial y}\frac{h_1'}{H}. \tag{18}$$