The Relationship between Water Mass Formation and the Surface Buoyancy Flux, with Application to Phillips' Red Sea Model

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ABSTRACT

A buoyancy flux across the sea surface between the outcropping of isopycnals must be balanced by a subsurface diapycnal buoyancy flux. If this flux were only advective, its derivative with respect to buoyancy would provide a direct estimate of the buildup of volume between isopycnals or rate of water mass formation. A diapycnal velocity, however, requires diapycnal mixing which also causes a diapycnal buoyancy flux, and it is shown that there is no reason to expect a simple relationship between the advective and diffusive fluxes. For a surface layer with vigorous vertical mixing and weak horizontal mixing, however, the diapycnal diffusive flux of buoyancy is small, and the flow through the base of the mixed layer can be derived from the derivative of the surface buoyancy flux with respect to buoyancy. These points are illustrated by examination of the Phillips similarity solution for the convective circulation driven in a channel by a uniform surface buoyancy loss.

1. Introduction

An intriguing way of looking at water mass formation and ocean circulation was introduced by Walin (1982) and pursued by Tziperman (1986) and Speer (1993). Figure 1 is a schematic section of the ocean showing two isopycnal surfaces, with buoyancy b and $b + \delta b$ [where $b = -g(\rho - \rho_0)/\rho_0$, with ρ_0 a reference density], intersecting the sea surface. It is supposed that the area of the sea surface between the outcropping of these two isopycnals is δS and that the spatially averaged surface buoyancy loss over this area is B_0 per unit area [given by $B_0 = -c_w^{-1} g\alpha\rho_0^{-1} Q + g\beta s\rho_0^{-1}(E - P)$, where c_w is the specific heat of water at constant pressure, $\alpha = -\rho^{-1}(\partial\rho/\partial T)_{p,s}$ is the coefficient of expansion of water at fixed pressure p and salinity s, Q is the net heat flux, $\beta = \rho^{-1}(\partial\rho/\partial s)_{p,T}$, E is the evaporation rate, and P is the rainfall rate (Gill 1982)]. However, by the definition of an isopycnal, the density of the fluid between these isopycnals cannot change, so the surface buoyancy loss must be balanced within the ocean by a convergence of the buoyancy flux across the two isopycnals. The problem need not be steady with stationary isopycnals, though for the present we assume that it is and return to the issue of time dependence in section 2b.

Let $A(b)$ represent the *volume* flux across the isopycnal with buoyancy b between the sea surface and a control surface C (Fig. 1). The outward volume flux across C between b and $b + \delta b$ is then $-(dA/db)\delta b$. The advective *buoyancy* flux across b is Ab, and we denote the diffusive buoyancy flux by

$$D = \int_b -K_d \frac{\partial b}{\partial n} dS_b, \qquad (1.1)$$

where K_d is the diapycnal eddy diffusivity and n a coordinate normal to isopycnals (increasing in the direction of increasing b). The integral is over the isopycnal surface in the region of interest, with dS_b an area element of that surface. (In the purely laminar case the diapycnal diffusive flux is zero in view of the definition of velocity as mass flux divided by density, but we assume that A refers to a mean flow and that there are unresolved motions that are parameterized with the eddy diffusivity K_d.)

Considering the control volume between the sea surface and surface C, and between isopycnals b and $b + \delta b$, in the limit $\delta b \to 0$ (Fig. 1) the buoyancy budget implies

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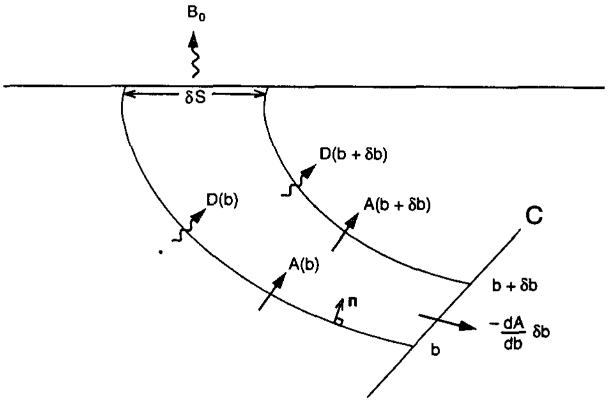

FIG. 1. Schematic of the surface outcropping of isopycnals with buoyancy b and $b + \delta b$. The sea surface area between the two isopycnals is δS, and the spatially averaged rate of surface buoyancy loss there is B_0. The location of the bounding control surface C is arbitrary.

$$\frac{d}{db}(Ab) + B_0\frac{dS}{db} + \frac{dD}{db} - b\frac{dA}{db} = 0, \quad (1.2)$$

where a steady state is assumed and there is no contribution from an expanding or contracting control volume. Rearranging,

$$A = -B_0(dS/db) - dD/db. \quad (1.3)$$

If we define F in terms of the surface forcing as

$$F = -B_0(dS/db), \quad (1.4)$$

then the diapycnal volume flux A differs from F by a term $-dD/db$.

The subduction across the surface C, or water mass formation rate $-dA/db$, can be written as

$$-dA/db = -dF/db + d^2D/db^2. \quad (1.5)$$

The first term on the right-hand side is derivable from surface properties and can be regarded as the formation rate associated with air–sea interaction. It is associated with a flow A across isopycnals, but this diapycnal advection can only occur in the presence of diapycnal mixing; otherwise the isopycnal would just be advected. This mixing then also contributes to the diapycnal buoyancy flux, and $-dF/db$ cannot be interpreted as the mean outflow.

The contribution of both diapycnal advection and mixing was recognized by Walin (1982) (who, incidentally, examined the problem in terms of temperature rather than buoyancy), though he did not quantify the balance between advection, mixing, and heating, and his main application assumed that the diffusive flux was negligible. Tziperman (1986) erroneously stated that the only diapycnal mass flux is advective, though he did recognize that diapycnal advection requires mixing. Speer and Tziperman (1992) computed F and dF/db for the North Atlantic and also mentioned the role of interior mixing. The interpretation of the functions F and dF/db in terms of interior flow and mixing thus remains rather unclear; there appears to be room for clarification of the problem and an investigation of the relative importance of advection and mixing for particular choices of the control surface C.

The role of the diffusive flux is most obviously crucial if the region being examined is large enough for the isopycnal b to terminate everywhere at a solid boundary or the sea surface, enclosing a bowl of lighter water between it and the sea surface. In that case, as recognized by Walin (1982), the volume flux across b must be zero if the volume of the bowl above it is constant, so that the diapycnal buoyancy flux required to balance B_0 must be diffusive. In smaller regions, with exchanges permitted across sills or open boundaries, the advective flux does not cancel but determines the strength of the exchange. There is also the likelihood in time-dependent problems that the volume of water between two isopycnals changes, particularly seasonally, so that at any given time the surface buoyancy flux need not be balanced by diapycnal mixing even for an isopycnal below a closed bowl of fluid.

The purpose of this paper is to examine the relationship between advective and diffusive diapycnal buoyancy fluxes. General formulas (section 2) for an arbitrary choice of the control surface C will suggest that there is no simple relationship, so that dF/db with F from (1.4) must not be interpreted in general as a mean flow and does not, in fact, even give an upper or lower bound for the net water mass formation rate. It will be shown in section 2a, however, that if C is the base of a mixed layer, within which the isopycnals are nearly vertical, and if the mixing in this layer is only in the vertical direction, then the second term on the right-hand side of (1.5) is negligible and, at least for a steady state, the outflow from the base of the mixed layer is given by $-dF/db$. This is not surprising, but seems not to have been explicitly discussed before, although the connection between F and mixed layer flow was noted by Speer (1993).

In section 3 we verify these general conclusions for similarity solutions of Phillips' (1966) Red Sea model, a two-dimensional (x, z) channel of uniform depth driven by a steady, uniform, surface buoyancy loss. Particular solutions of this (section 4) show that, as expected, there is generally no simple relationship between A and F, or dA/db (representing outflow as a function of b) and dF/db, if C is a vertical control surface at the open end of the channel. In special circumstances, however, the diffusive diapycnal flux may be negligible, so that $-dF/db$ does become a direct and useful measure of water mass formation, and it is also confirmed that this is true if C is taken to be the base of a surface layer of strong mixing. These results together with other aspects of the problem are reviewed in section 5.

2. Advection or diffusion?

The volume flux A may be written as

$$A = \int_b (\mathbf{u}\cdot\mathbf{n})dS_b\,, \tag{2.1}$$

where $\mathbf{n}$ is the unit normal to the isopycnal and, ignoring minor metric changes associated with an isopycnal/diapycnal coordinate system, $\mathbf{u}$ satisfies the buoyancy equation

$$\mathbf{u}\cdot\nabla b = \frac{\partial}{\partial n}\left(K_d \frac{\partial b}{\partial n}\right) \tag{2.2}$$

for a steady state (we discuss time dependence later). After substituting $\nabla b = (\partial b/\partial n)\mathbf{n}$, the advective buoyancy flux may be written

$$Ab = \int_b \left[b\frac{\partial}{\partial n}\left(K_d \frac{\partial b}{\partial n}\right) \Big/ \left(\frac{\partial b}{\partial n}\right)\right] dS_b\,. \tag{2.3}$$

The contributions Ab from (2.3) and D from (1.1) show similar combinations of b, K_d and n, but there appears to be no reason to expect any simple relationship between them. The contribution Ab from (2.3) is also sensitive to the choice of the reference density ρ_0, so it is appropriate instead to examine the divergence of the buoyancy flux and compare the different terms in (1.5).

From (2.3) we may write

$$A = \int_b \frac{\partial}{\partial b}\left(K_d \frac{\partial b}{\partial n}\right) dS_b\,, \tag{2.4}$$

whereas

$$-dD/db = \frac{d}{db}\int_b K_d \frac{\partial b}{\partial n} dS_b \tag{2.5}$$

and we cannot take the derivative d/db inside the integral to operate solely on $K_d\partial b/\partial n$ since the lateral boundaries of the control volume, and particularly that at the sea surface, are not orthogonal to isopycnals. [If the lateral boundaries of the control volume were orthogonal to isopycnals, we could interchange the derivative and integral in (2.5) to obtain $A = -dD/db$, simply a derivation of (2.2)!]

There is no reason to expect a simple relationship between A and dD/db in general, so that deriving A, or the more important dA/db, in terms of $B_0(dS/db)$ does not seem possible without a knowledge of the mixing. One could still regard the net interior formation rate $-dA/db$ as being made up of a known formation rate

$$\frac{d}{db}\left(B_0 \frac{dS}{db}\right)$$

from air–sea interaction and a formation rate d^2D/db^2 from mixing, but the utility of calculating the former would seem to be limited unless something definite can be said about the latter. This does turn out to be possible if the control surface C is taken to be the base of a surface layer with strong vertical, but not horizontal, mixing. And, of course, as emphasized earlier, for a closed domain $A = 0$ so that the air–sea flux must be balanced by diapycnal mixing.

a. Scale analysis for a mixed layer

We can compare the two terms on the right-hand side of (1.3) for the situation where C is the base of a surface mixing layer of thickness h. If κ denotes the vertical eddy diffusivity in this layer, and we ignore horizontal mixing, then the diapycnal buoyancy flux is given by

$$D = -\int \kappa \frac{\partial b}{\partial z} dx\,, \tag{2.6}$$

where we take coordinates x horizontally and z vertically in Fig. 1 and are considering the problem per unit horizontal distance in the other horizontal direction. The integral in (2.6) is over the horizontal distance covered by the isopycnal of buoyancy b from the base of the surface layer to the sea surface. It may be written

$$D = \int_{-h}^{0} \kappa \frac{\partial b}{\partial z} s^{-1} dz\,, \tag{2.7}$$

where s is the isopycnal slope dz/dx. The z integral is for constant b rather than constant x. Given the surface boundary condition $\kappa\partial b/\partial z = -B_0$ at $z = 0$, D is thus of order $s^{-1}h\ B_0$, where a typical value of s in the surface layer is used. Hence, we can compare the mixing term of (1.3) with the surface buoyancy flux term, obtaining

$$\frac{|dD/db|}{|B_0(\partial x/\partial b)_{z=0}|} \approx \left| B_0^{-1}\frac{d}{dx}(s^{-1}hB_0)\right|. \tag{2.8}$$

If h is the only quantity that varies with x, then the ratio becomes

$$\left|\frac{dh}{dx}\Big/ s\right|$$

and is small if, as is likely, the slope of the mixed layer base is much less than that of isopycnals in the mixed layer. Alternatively, the ratio is of order $(h/L)s^{-1}$, where L is the horizontal scale of variation of B_0 or s.

In the next section we shall compare A and F for a particular circulation model, but first we consider the extension of the results so far to the time-dependent case.

b. Time dependence

The diapycnal velocity is now $(\mathbf{u} - \mathbf{U})\cdot\mathbf{n}$, where $\mathbf{U}$ is the motion of isopycnals relative to fixed coordinates and thus satisfies

$$\partial b/\partial t + \mathbf{U}\cdot\nabla b = 0\,, \tag{2.9}$$

or

$$\partial b/\partial t + \mathbf{U}\cdot\mathbf{n}(\partial b/\partial n) = 0 . \quad (2.10)$$

The buoyancy equation is now

$$\frac{\partial b}{\partial t} + \mathbf{u}\cdot\nabla b = \frac{\partial}{\partial n}\left(K_d \frac{\partial b}{\partial n}\right), \quad (2.11)$$

and again $\mathbf{u}\cdot\nabla b = (\mathbf{u}\cdot\mathbf{n})(\partial b/\partial n)$ so that subtracting (2.10) from (2.11) we have a diapycnal flow

$$(\mathbf{u} - \mathbf{U})\cdot\mathbf{n} = \frac{\partial}{\partial n}\left(K_d \frac{\partial b}{\partial n}\right) \Big/ \left(\frac{\partial b}{\partial n}\right), \quad (2.12)$$

exactly as for the steady problem.

The derivation of A, and hence the outflow $-(\partial A/\partial b)\delta b$ from the control volume, depends upon assuming no changes in the volume of the control volume. Assuming, though, that changes in this volume also count as water mass formation, (1.3) may still be used. This issue need not arise if one averages over a seasonal cycle, but evaluation of (1.3) would then require averaging the terms on the right-hand side for each b. As for the steady case, there is no reason to assume the diffusion term on the right-hand side to be either negligible or to be simply related to the other terms. It is not yet clear whether the result of section 2a can be carried over easily to the time-dependent problem.

3. Application to the Phillips Red Sea model

The results of the preceding sections suggest that there is no general relationship between the diapycnal advection $A(b)$ alone and the quantity $F = -B_0(dS/db)$ unless the control surface C is at the base of a surface layer with strong mixing. To illustrate this, and to check for particular situations in which A and F might be simply related, it seems worthwhile to evaluate both A and F for a model for which the solutions are known.

Here we consider the circulation in the vertical plane ($0 \leqslant x \leqslant L$, $0 \leqslant z \leqslant h$) of a narrow sea of length L driven by a constant and spatially uniform surface buoyancy loss rate B_0 and separated from the exterior ocean by a sill of depth h. With the Boussinesq approximation and retaining only vertical mixing, the governing equations are (Phillips 1966)

$$\mathbf{u}\cdot\nabla b = \frac{\partial}{\partial z}\left(\kappa \frac{\partial b}{\partial z}\right) \quad (3.1)$$

$$\mathbf{u}\cdot\nabla u + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = \frac{\partial}{\partial z}\left(\nu \frac{\partial u}{\partial z}\right) \quad (3.2)$$

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} = b \quad (3.3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 , \quad (3.4)$$

where the buoyancy $b = -g(\rho - \rho_0)/\rho_0$ is referenced to the density ρ_0 of the assumed stagnant reservoir at $z < 0$, the perturbation pressure p' is referenced to the pressure in a fluid at rest with density ρ_0, and we have assumed a hydrostatic balance in the vertical (as appropriate for $h \ll L$).

Phillips argues that a solution independent of L is given by the similarity forms

$$b = (B_0 x)^{2/3} h^{-1} g(\eta) \quad (3.5)$$

$$u = (B_0 x)^{1/3} f(\eta) , \quad (3.6)$$

where $\eta = z/h$. The exponents of x are chosen to satisfy the buoyancy integral

$$\int_0^h ub\,dz = -B_0 x \quad (3.7)$$

and to give the same x dependence of the advective and buoyancy torque terms in the vorticity equation

$$\frac{\partial}{\partial z}\left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right) = \frac{\partial^2}{\partial z^2}\left(\nu \frac{\partial u}{\partial z}\right) - \frac{\partial b}{\partial x} . \quad (3.8)$$

We introduce a streamfunction $\psi(\eta)$ such that $f = d\psi/d\eta$, and may then write the vertical velocity as

$$w = -\frac{1}{3} B_0^{1/3} x^{-2/3} h \psi , \quad (3.9)$$

showing an integrable singularity at $x = 0$.

For the diffusive and viscous terms in (3.1) and (3.8) to have the same x dependence as the other terms, we require the eddy coefficients to be of the form

$$\kappa = \frac{1}{3}(B_0/x^2)^{1/3} h^2 K(\eta),$$

$$\nu = \frac{1}{3}(B_0/x^2)^{1/3} h^2 N(\eta) , \quad (3.10)$$

also increasing toward the head of the sea. The buoyancy and velocity equations then become the nonlinear coupled ordinary differential equations

$$2\psi' g - \psi g' = (K g')' \quad (3.11)$$

$$(\psi'^2 - \psi\psi'')' = (N\psi'')'' - 2g , \quad (3.12)$$

where the prime denotes $d/d\eta$. These may be solved numerically by decomposition into the six first-order differential equations

$$\psi' = q_1 \quad (3.13)$$

$$g' = q_2/K \quad (3.14)$$

$$q_1' = q_3/N \quad (3.15)$$

$$q_2' = 2q_1 g - \psi q_2/K \quad (3.16)$$

$$q_3' = q_4 + q_1^2 - \psi q_3/N \quad (3.17)$$

$$q_4' = 2g, \quad (3.18)$$

where (3.13), (3.14), (3.15), (3.17) define four new functions q_1, q_2, q_3, q_4, respectively.

Two boundary conditions at the top ($\eta = 1$) and bottom ($\eta = 0$) are

$$\psi = 0 \quad \text{at} \quad \eta = 0, 1 \,, \tag{3.19}$$

and zero stress at the top and bottom gives two more as

$$q_3 = 0 \quad \text{at} \quad \eta = 0, 1 \,. \tag{3.20}$$

The surface buoyancy flux condition $-\kappa\, \partial b/\partial z = B_0$ at $z = h$ becomes

$$q_2 = -3 \quad \text{at} \quad \eta = 1 \,. \tag{3.21}$$

Only one more boundary condition is allowable, though for the problem as posed we would like to have both $b = 0$ at $z = 0$ so that $g(0) = 0$, and zero buoyancy flux $Kg' = 0$ at $\eta = 0$ so that the flow is driven only at the surface. We can achieve this by choosing $K(0) = 0$ without an extra condition on $g'(0)$. In fact, if we assume that both ψ and g are proportional to η as $\eta \to 0$, then (3.11) can only be in balance as $\eta \to 0$ if $Kg' \propto \eta^2$, which implies that K has to be proportional to η^2. Also, allowing N to tend to 0 as $\eta \to 0$ would make the problem degenerate, so for the moment we keep N finite. Subject to these constraints we may solve for $\psi(\eta)$ and $g(\eta)$ for arbitrary $K(\eta)$ and $N(\eta)$.

The water mass transformation function $F(b)$ may be written

$$F(b) = -B_0(\partial x/\partial b)_{z=h} \tag{3.22}$$

$$= -\frac{3}{2}\left(\frac{b}{b_0}\right)^{1/2}(B_0L)^{1/3}\frac{h}{g(1)}\,, \tag{3.23}$$

whereas, with the control surface C taken as the vertical line at $x = L$, the actual advective flow across an isopycnal with buoyancy b is

$$A(b) = \int_{z_b}^{h} u(L, z)\, dz = -\int_0^{z_b} u(L, z)\, dz\,, \tag{3.24}$$

where z_b is the depth of the isopycnal at $x = L$. From (3.6) and the definition of ψ,

$$A(b) = -(B_0L)^{1/3}h\psi(\eta_b), \quad \text{where} \quad \eta_b = z_b/h\,. \tag{3.25}$$

The ratio of actual advection to water mass transformation is

$$A/F = \frac{2}{3}\left(\frac{b}{b_0}\right)^{-1/2} g(1)\psi(\eta_b)$$

$$= \frac{2}{3}\left(\frac{g(1)}{g}\right)^{1/2} g(1)\psi(\eta_b)\,, \tag{3.26}$$

which does not have any general interpretation, although we note that the constraint (3.7) becomes

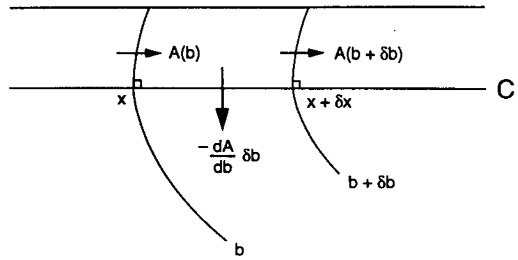

FIG. 2. The control surface C is chosen to be at the base of the surface layer, defined by the depth at which the isopycnals are vertical.

$$\int_0^1 (d\psi/d\eta) g\, d\eta = -1\,. \tag{3.27}$$

A key function of interest is the ratio of the net outflow rate $-dA/db$ to the known function $-dF/db$. This ratio is given by

$$\left(\frac{dA}{db}\right)\bigg/\left(\frac{dF}{db}\right) = \frac{4}{3}\left(\frac{d\psi}{d\eta}\right)\left(\frac{dg}{d\eta}\right)^{-1}[g(1)]^{3/2}g^{1/2}, \tag{3.28}$$

with g and the derivatives of ψ and g evaluated at $\eta = \eta_b$.

We can also contrast the maximum inflow with the maximum transformation rate, which is that for b_0 (although there are subsurface values of the buoyancy greater than b_0 due to hydrostatic instability of the upper part of the water column). This ratio is given by

$$A_{\max}/F_{\max} = \frac{2}{3}g(1)\psi_{\max}\,. \tag{3.29}$$

An upper bound for this ratio is obtained by noting that $\psi(0) = \psi(1) = 0$ and assuming that ψ has just one maximum. Then the constraint (3.27) gives

$$\int_0^{\psi_{\max}} (g_{\text{above}} - g_{\text{below}})\, d\psi = 1\,, \tag{3.30}$$

where g_{above} and g_{below} refer to values of g above and below the level where $\psi = \psi_{\max}$. Given that g_{below} is positive and g_{above} in a well-mixed surface layer does not greatly exceed the surface value $g(1)$, we have an approximate lower bound of 1 for $g(1)\psi_{\max}$ and hence $A_{\max} \geqslant \frac{2}{3}F_{\max}$. This lower bound for $A_{\max}$ shows that the total inflow cannot be much smaller than the total transformation, though it can be greater if strong interior mixing exists, effectively reducing the vertical density contrast.

One final parameter of interest in the solutions is the Richardson number. As shown by Phillips (1966), it is independent of horizontal position and is given as a function of depth by $\text{Ri} = (dg/d\eta)(d^2\psi/d\eta^2)^{-2}$.

Flux out of the surface layer

So far we have taken the control surface C to be the vertical line at $x = L$. Following the discussion of sec-

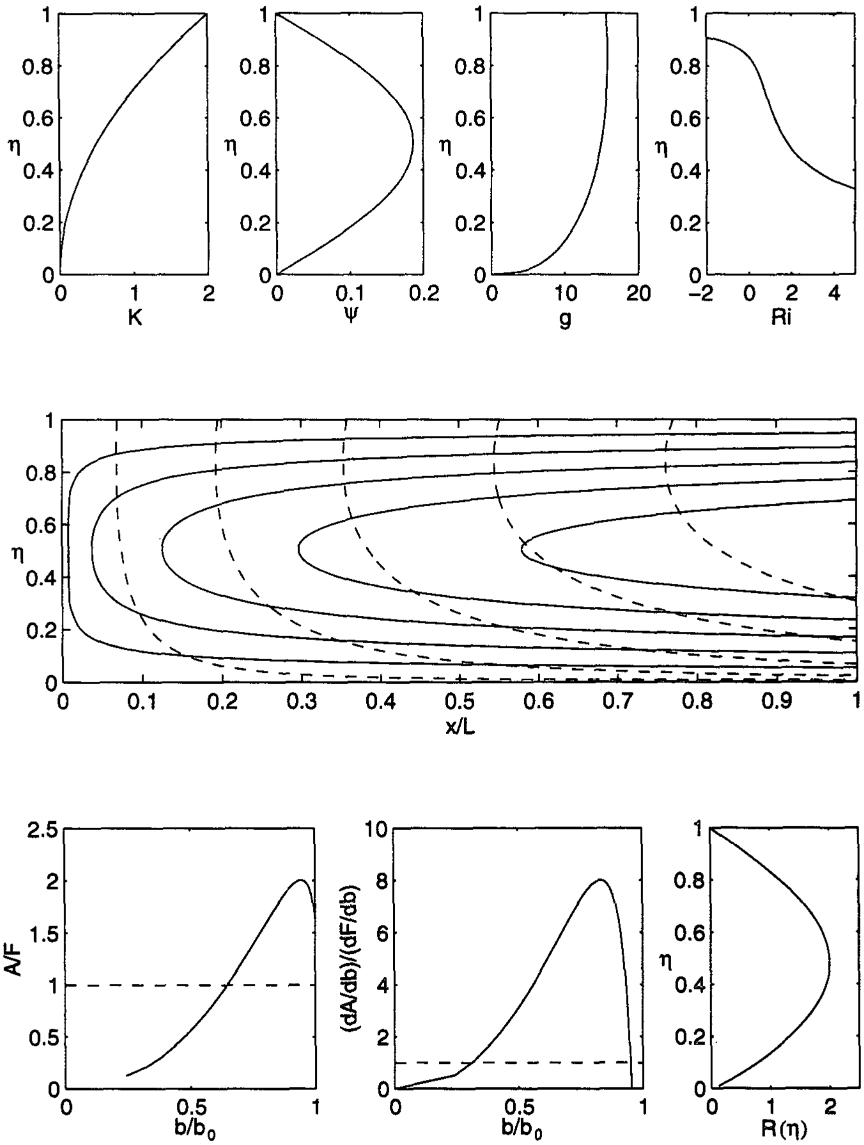

FIG. 3. Solution of the Phillips (1966) problem for $K = 2\eta^2$ and $N = 2$. The first three panels in the top row show the profiles for the similarity functions describing the eddy diffusivity, streamfunction, and buoyancy; the fourth panel shows the Richardson number of the flow. We also show the streamlines (solid) and isopycnals (dashed) for the solution in x, z space. The isopycnals are for six equally spaced values of b in the range from 0 to b_0, and the streamlines similarly for ψ between 0 and ψ_{max}. The first two panels in the bottom row show, as functions of the scaled buoyancy b/b_0, the ratios A/F of diapycnal advection A to the sea surface transformation function $F = -B_0(dS/dB)$ and of the actual water mass formation rate dA/dB to dF/dB (both for a vertical control surface at $x = L$). The final panel shows the function $R(\eta)$ from (3.31), which gives both ratios for a control surface C chosen to be horizontal at depth η.

tion 2, we also consider the consequences of taking C to be horizontal, particularly at the base of any surface layer with strong mixing. In this case, $A(b)$ is the diapycnal volume flux in the surface layer, and the outflow $-dA/db$ from the surface layer across the level η is $-w dx/db$. Using (3.9) and (3.5), and with dF/db from (3.22), we obtain the ratio of interest

$$R(\eta) = \frac{dA/db}{dF/db} = \frac{2}{3}\psi(\eta)g(1)^{3/2}g(\eta)^{-1/2}, \quad (3.31)$$

which may be evaluated for any η, but particularly for $\eta = \eta_m$ corresponding to the base of a surface layer of strong mixing in which case the isopycnals are steep in the slightly unstable mixed layer and we may define

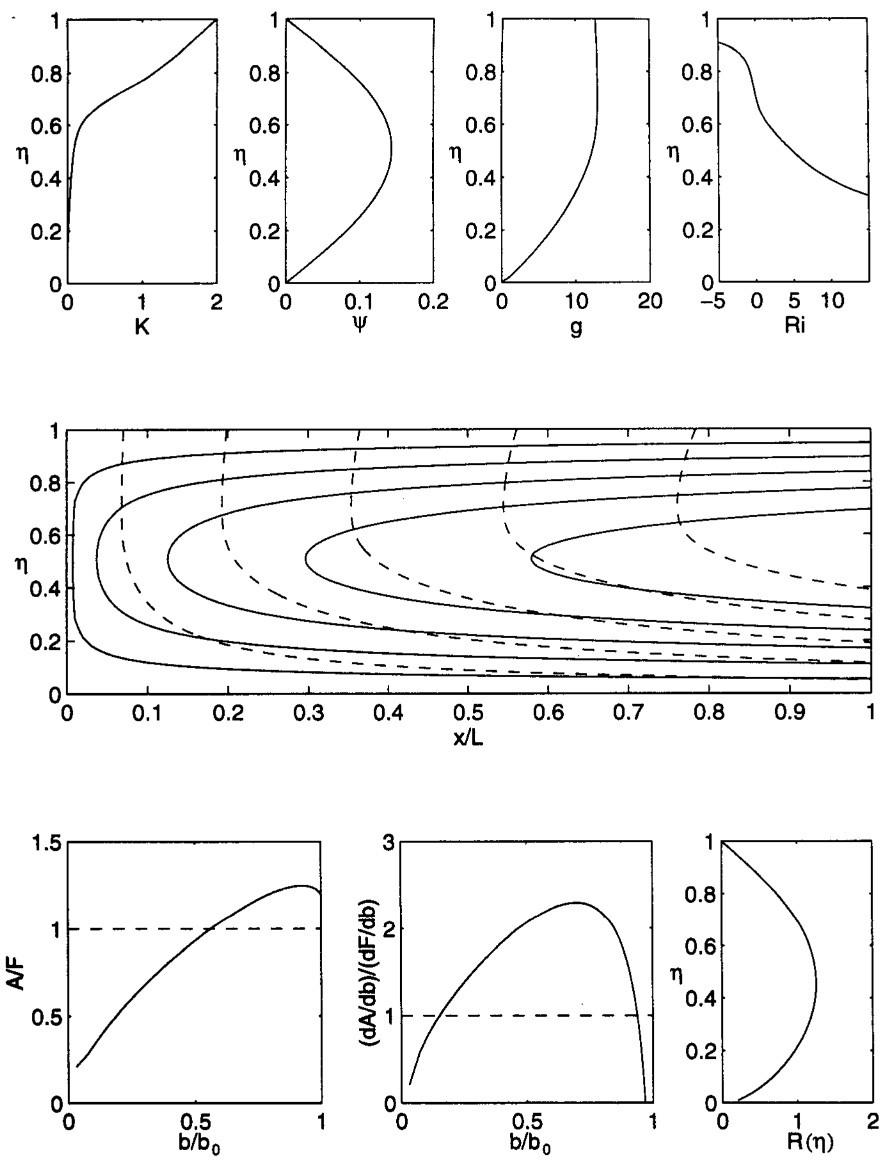

FIG. 4. As in Fig. 3 but with K given by (4.1) and $N = 2$.

η_m as the depth at which the isopycnals become vertical (Fig. 2). For any η the same ratio $R(\eta)$ also gives the value of A/F since $A = F = 0$ for $b = 0$.

4. Particular solutions

The system of equations (3.13)–(3.18) is solved for the six functions ψ, g, q_1, q_2, q_3, q_4 as an ordinary two-point boundary value problem. We do this by iterating with the Newton–Raphson method, which implements a fourth-order Runge–Kutta scheme (Press et al. 1986). As shown earlier, $K(\eta) \propto \eta^2$ close to the bottom. We thus solve the problem first with $K = c\eta^2$ for all η, where c is an arbitrary constant. For convenience we also take the viscosity function N to be constant independent of η. Figure 3 shows the solution for $c = 2$ and $N = 2$. The profiles, and the corresponding circulation and isopycnals for the similarity solution, show that, as expected, water enters the basin near the surface, loses buoyancy, and leaves the basin at depth. Due to the surface buoyancy loss, there is a hydrostatically unstable layer below the surface, and flow with Ri < 1/4 persists in a thin layer below this. We also show A/F and $(dA/db)/(dF/db)$ as functions of b; these confirm that there is no simple relationship between A and F. The ratio $A_{\max}/F_{\max}$ from (3.29) is 1.95. The function $R(\eta)$ from (3.31) is close to 1 at the base of the layer of steep isopycnals, as expected from the argument of section 2a, with a value of 0.99 at the level, $\eta_m = 0.83$, where $dg/d\eta = 0$

and the vertical buoyancy gradient changes from unstable above to stable below. For the chosen profile of eddy diffusivity, however, sufficient mixing occurs below η_m to make $R(\eta)$ there significantly different from 1.

We next modify the eddy diffusivity profile to

$$K(\eta) = \eta^2\left[\frac{1}{2}(K_0 + K_1) + \frac{1}{2}(K_0 - K_1)\tanh(\eta - H)/\epsilon\right] \quad (4.1)$$

in order to represent a near-surface layer of strong mixing K_0 and much weaker mixing K_1 close to the bottom. The transition layer between the two regions occurs at $\eta = H$ and has thickness ϵ. The results for this profile with $K_0 = 2$, $K_1 = 0.3$, $\epsilon = 0.1$, $H = 0.7$, and $N = 2$ are shown in Fig. 4. The flow is qualitatively as in the earlier example, but with a reasonable match of strong mixing for negative and small positive values of the Richardson number, and weak mixing for Ri $> 1/4$. For this case $A_{\max}/F_{\max} = 1.22$, and again there is no simple relationship between the diapycnal advection A and the water mass transformation rate F. The changes from the first case illustrate the sensitivity to the diffusivity profile, but we note that, with the mixing largely confined to a surface layer, the ratio A/F is closer to 1 than for the previous case with more mixing at depth. Indeed $R(\eta)$ from (3.31) now achieves a value of 0.98 at $\eta_m = 0.71$, where $dg/d\eta = 0$, and stays close to 1 until close to the bottom where mixing occurs over a sufficiently large area to cause significant flow across isopycnals. This all suggests that it would be worthwhile to examine the solution for the situation in which the deep mixing is zero.

A well-mixed layer above a perfect fluid

We consider the case of a well-mixed layer above a perfect fluid. We assume that the top layer, with streamfunction ψ_1 and buoyancy function g_1, extends from $\eta = H$ to $\eta = 1$ and has large vertical eddy diffusivity and viscosity functions K and N, whereas the lower layer from $\eta = 0$ to $\eta = H$, with ψ_2 and g_2, has $K = N = 0$.

The boundary conditions are

$$\psi_1 = \psi_1'' = 0, \quad Kg_1' = -3 \quad \text{at} \quad \eta = 1 \quad (4.2)$$

and

$$\psi_2 = g_2 = 0 \quad \text{at} \quad \eta = 0\,. \quad (4.3)$$

As $N = 0$, zero stress at $\eta = 0$ is satisfied without a requirement for $\psi_2'' = 0$. The problem thus appears to be degenerate, with a single infinity of possible solutions. The matching conditions at $\eta = H$ are

$$\psi_1 = \psi_2, \quad g_1 = g_2, \quad \psi_1' = \psi_2',$$
$$Kg_1' = N\psi_1'' = 0, \quad N\psi_1''' = \psi_2\psi_2''\,. \quad (4.4)$$

For sufficiently large N, (3.12) may be approximated by $N\psi_1'''' = 0$. Hence, to satisfy $\psi_1 = \psi_1'' = 0$ at $\eta = 1$,

$$\psi_1 = a_1(1 - \eta) + O[(1 - \eta)^3] \quad (4.5)$$

in which the fifth matching condition at $\eta = H$ shows that the second term is negligible. Neglecting the second term on the left-hand side of the buoyancy equation (3.11), as will be justified shortly, we have

$$Kg_1'' = 2\psi_1' g_1 \quad (4.6)$$

with solution

$$g_1 = a_2\cos[(2a_1/K)^{1/2}(\eta - H)] \quad (4.7)$$

in order to satisfy $g_1' = 0$ at $\eta = 1$. In addition to large N, the neglect of $\psi g'$ in (3.11) requires $(a_1/K)^{1/2}(1 - H) \ll 1$, which we assume is satisfied. The surface boundary condition $Kg' = -3$ at $\eta = H$ then implies

$$2a_1a_2(1 - H) = 3\,. \quad (4.8)$$

We thus have a slablike mixed layer moving with uniform speed and with a density approximately independent of η.

In the lower layer, the buoyancy equation (3.11) with $K = 0$ implies $g_2 \propto \psi_2^2$, so that buoyancy is conserved on streamlines. Matching to surface layer values implies

$$g = a_2(\psi/\psi_H)^2\,, \quad (4.9)$$

where $\psi_H = a_1(1 - H)$ is $\psi_1(H)$. Now from (4.9) and (4.7), $\psi(\eta_b) = \psi_H(g/g(1))^{1/2}$, so from (3.26) $A = F$ for all b, and hence $dA/db = dF/db$ also. Moreover, for this case the function $R(\eta)$ from (3.31) increases from 0 at the surface to 1 at the base of the mixed layer and stays equal to 1 for all other values of η.

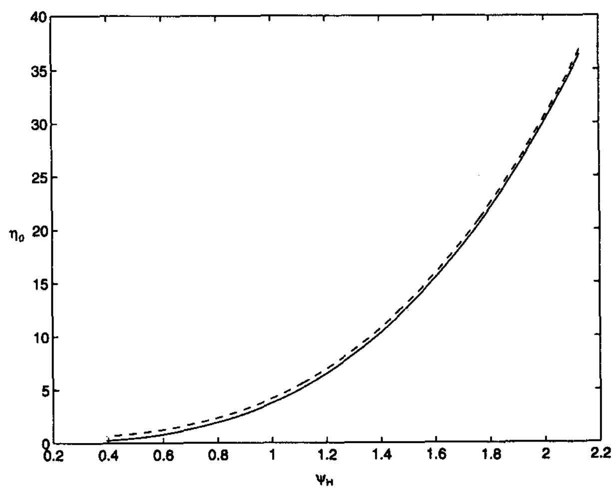

FIG. 5. The exact (solid) and asymptotic (dashed) relationships between the scaled streamfunction ψ_H at the base of a well-mixed layer and the reference level η_0 in the solution for inviscid adiabatic flow beneath the mixed layer.

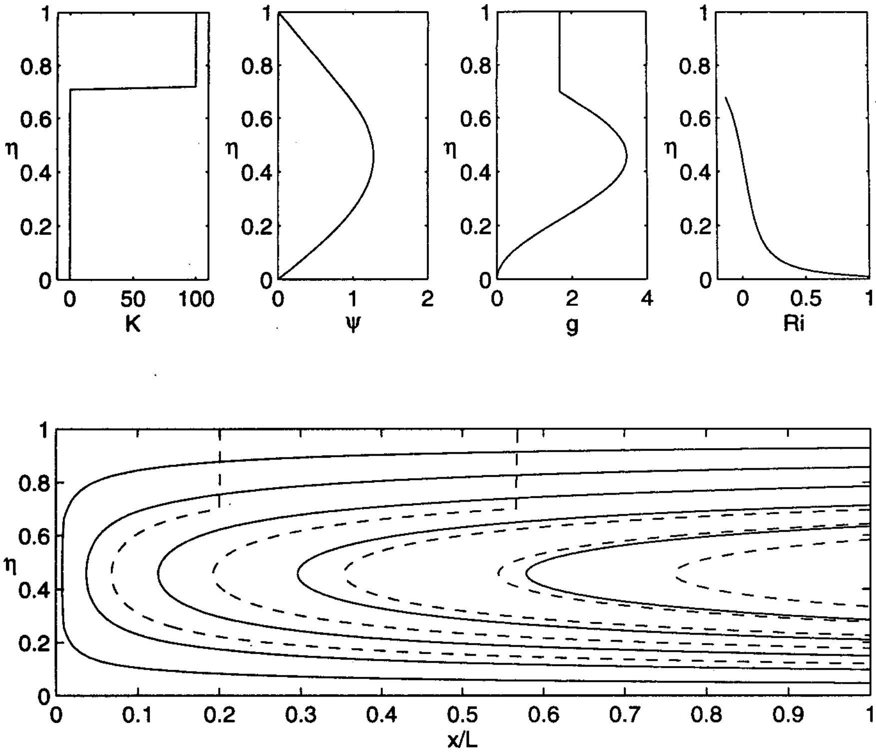

FIG. 6. As in Fig. 3 but for the model of a well-mixed layer above a perfect fluid. [The bottom row of panels is omitted since in this case $A = F$ and so also $dA/db = dF/db$. $R(\eta)$ simply increases from 0 at the sea surface to 1 at the base of the mixed layer, then stays equal to 1 below this.]

The analytical solution to the problem thus shows the physically plausible picture of water entering the system and having its buoyancy changed by surface buoyancy loss and vigorous vertical mixing before emerging from the mixed layer and leaving the system adiabatically. The result $A = F$ depends on the neglect of horizontal mixing in the mixed layer as well as on strong vertical mixing.

The vorticity equation (3.12) in the lower layer becomes

$$\psi_2\psi_2''' - \psi_2'\psi_2'' = 2g = 2a_2\psi_H^{-2}\psi_2^2 \qquad (4.10)$$

or, using (4.9),

$$(\psi_2''/\psi_2)' = 3\psi_H^{-3}, \qquad (4.11)$$

leading to Airy's equation

$$\psi_2'' = 3\psi_H^{-3}(\eta - \eta_0)\psi_2 \qquad (4.12)$$

with solutions

$$\psi_2 = a_3 Ai[3^{1/3}\psi_H^{-1}(\eta - \eta_0)] + a_4 Bi[3^{1/3}\psi_H^{-1}(\eta - \eta_0)]\,. \qquad (4.13)$$

Matching conditions at the interface at $\eta = H$ require continuity of ψ and ψ' and of expressions involving higher derivatives of ψ and hence affecting the small higher-order terms in ψ_1. In the absence of viscosity in the lower layer, we only have $\psi = 0$ at $\eta = 0$, and hence only three boundary conditions for the four unknowns a_3, a_4, η_0, and ψ_H. The problem has thus become degenerate, with solutions existing for any choice of ψ_H.

In fact, solutions only exist for $\psi_H > 0$ and then imply $\eta_0 > 0$. The relationship between η_0 and ψ_H is shown in Fig. 5 for $H = 0.7$, together with the formula $\eta_0 = 3.76\psi_H^3$ that is asymptotically valid then for large η_0 using asymptotic formulas for the Airy functions. Figure 6 shows the solution for $\psi_H = 0.89$ and with $K = 100$ in the surface mixed layer. In this case $A_{max}/F_{max} = 1.43$. Physically unrealistic static instability ($g' < 0$) without mixing occurs below the mixed layer as water is expelled from it with negative x momentum and continues toward the head of the sea before buoyancy torques turn it around. This is a weakness of the present model with the assumption that all mixing takes place in a vigorous surface layer above a fixed level with no mixing below it, but presumably need not occur in a three-dimensional situation.

5. Discussion

Analysis of the buoyancy equations and the solutions of a particular model have shown that there is no direct general connection between water mass transformation, due to a surface buoyancy flux, and the production of a particular water mass. However, the model does confirm what is clearly a general result, that the water mass formation rate can be determined from the surface buoyancy flux if horizontal mixing is ignored and if the vertical mixing is strong in a surface layer, with very nearly vertical isopycnals, and zero in the ocean interior below the surface layer. In this case the diapycnal advection A is given by the function F, defined by $F = -B_0(dS/db)$, which can be calculated from the surface buoyancy flux B_0 and the sea surface area δS between surface outcroppings of isopycnals with buoyancy b and $b + \delta b$, and the water mass formation rate $-dA/db$ is given by $-dF/db$. In all the cases examined, the outflow from the base of the surface layer is given to a good approximation by $-dF/db$, with F from (1.4).

This is probably still correct for the time-dependent problem of a deepening mixed layer above a perfect ocean interior (though more analysis is required). Hence the wintertime formation rate of, for example, 18° water, may be calculable from dF/db. It is not clear that the same interpretation is true during springtime shallowing of the mixed layer. In fact, for isopycnals that terminate everywhere at the free surface and enclose a bowl of fluid, the annual average of the advection A must be zero, so that the water mass transformation rate F must be balanced by mixing below the sea surface. It is possible that this balance is achieved by horizontal (and hence diapycnal) mixing in the surface layer, or perhaps is associated with seasonal changes in mixed layer depth. These effects require further theoretical investigation and application to the ocean and may show that the surface buoyancy flux does not require diapycnal mixing in the ocean interior below the base of the winter mixed layer.

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