A simple model of the hurricane boundary layer revisited

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Abstract:
A simple slab model for the boundary layer of a hurricane is reexamined and a small error in the original calculation is corrected. With this correction, the development of supergradient winds is a ubiquitous feature of the solutions. The boundary layer shows two types of behaviour in the inner core of the vortex depending on the depth of the layer and the maximum tangential wind speed above the layer. For small depths and/or large tangential wind speeds, large supergradient winds develop and lead to a rapid deceleration of the inflow such that the inflow becomes zero at some radius inside the radius of maximum tangential wind speed above the boundary layer. For large depths and/or small tangential wind speeds, the solutions do not become singular until within a few kilometres of the transition between the two regimes is very abrupt. Interpretations are given for the foregoing behaviour. Other aspects of the boundary-layer dynamics and thermodynamics are investigated including: the dependence on mixing by shallow convection; the effects of a radially-varying boundary-layer depth; the effects of downward momentum transport; the dependence of thermodynamical quantities on the boundary-layer depth; and the radial variation of equivalent potential temperature. Predicted values of the last quantity are in acceptable agreement with observations made in Category-5 Hurricane Isabel (2003). The version with radially-varying depth gives more realistic vertical velocities in the inner core region of the vortex. The limitation and strengths of the slab model are discussed.

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1 Introduction

The boundary layer is an important feature of a mature hurricane as it strongly constrains the radial distribution of vertical motion at its top as well as those of absolute angular momentum and moisture. Over the years it has been the subject of numerous theoretical investigations, many of them relating to axisymmetric vortices (Rosenthal 1962, Miller 1965, Smith 1968, Leslie and Smith 1970, Carrier 1971, Eliassen 1971, Bode and Smith 1975, Eliassen and Lystad 1977, Shapiro 1983, Montgomery et al. 2001, Smith 2003) and a few to asymmetric vortices (Shapiro 1983, Kepert 2001, Kepert and Wang 2001, Kepert 2006a, b). With the exception of Smith (2003, henceforth S03), these studies focussed exclusively on the dynamical constraints of the boundary layer. The importance also of the thermodynamical constraint was recognized by Emanuel (1986) and its representation was a key feature in the simple axisymmetric model he proposed for a mature hurricane. In that model, the tangential wind field above the boundary layer is assumed to be in thermal wind balance and air parcels flowing upwards and outwards into the upper troposphere are assumed to conserve their absolute angular momentum and moist entropy. The model is closed by a simple, uniform-depth slab formulation for the boundary layer, which is used to determine a functional relationship between the absolute angular momentum and moist entropy of air parcels leaving the boundary layer.

In S03 the first author presented a slightly more sophisticated model for the hurricane boundary layer than that employed by Emanuel op. cit., allowing for the effects of gradient wind imbalance, mean subsidence at large radii, and for the effects of shallow convection, which have an important control on the radial variation of thermodynamic quantities (without a representation of mixing by shallow convection, the boundary layer saturates at an unrealistically-large radius). Again the model was a steady, moist, axisymmetric, slab model of constant depth, but the tangential wind speed at the top of the layer was prescribed as a function of radius. With these assumptions the boundary layer equations reduce to a set of coupled ordinary differential equations for the radial variation of the boundary-layer wind, temperature and moisture fields. High-resolution numerical solutions of these equations were obtained by integrating inwards from some large radius, where it is assumed that geostrophic balance and convective-radiative equilibrium conditions prevail in the boundary layer. The model was used to explore various aspects of the boundary layer including the influence of vortex size and structure on the radial distribution of key dynamic and thermodynamic quantities. In particular it was shown that in some circumstances supergradient winds may develop in the boundary layer near the radius of maximum tangential wind speeds, a feature that has been found also in other studies (Eliassen and Lystad 1977, Shapiro 1983, Kepert 2001, Kepert and Wang 2001,
Recently it was discovered that the Runge-Kutta subroutine used to integrate the equations contained an error in the coefficients and, of course, this affects the solutions.

In the course of rectifying this error and investigating its implications, we discovered interesting new features of the boundary layer dynamics in the model, including the occurrence of an abrupt qualitative change in the character of the solutions when the boundary-layer depth exceeds a certain threshold value (changes in boundary layer depth were not explored in S03). We believe that these findings are worth reporting and such is one purpose of the present paper. The new calculations led us to examine the effects of allowing the boundary-layer depth to vary with radius and to reappraise the assumptions of boundary layer theory in the inner core region of the vortex where the flow is upwards out of the boundary layer. These aspects are discussed also.

This paper is organized as follows. The formulation of the model is reviewed briefly in section 2. We examine the boundary layer dynamics in the model, including a newly discovered sensitivity of the solutions in the inner core region of the vortex where the flow is upwards out of the boundary layer. These aspects are discussed also.

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2 Summary of the model

2.1 Boundary layer equations

We consider the boundary layer of a steady axisymmetric hurricane-like vortex on an f-plane. The boundary layer is assumed to have uniform depth \( \delta \) and constant density. In a cylindrical coordinate system \((r, \phi, z)\), the vertically-integrated equations for radial momentum, azimuthal momentum, heat or moisture, and continuity can be written in the form:

\[
\frac{du_b}{dr} = \frac{u_b}{\delta} \left( \frac{v^2}{gr} - \frac{v_b^2}{r} \right) - \frac{f(v_{gr} - v_b)}{\delta} - \frac{C_D}{\delta} \left( u_b^2 + v_b^2 \right) \frac{\delta}{\delta} u_b - \frac{\left( u'w' \right)_b}{\delta},
\]

\( \frac{dv_b}{dr} = \frac{u_b}{\delta} \left( v_b - v_{gr} \right) - \frac{\left( v_b - v_{gr} \right)}{r} + f u_b \)

\[
\frac{dw_b}{dr} = \frac{w_b}{\delta} \left( \chi_b - \chi_{b+} \right) + \frac{C_D}{\delta} \left( u_b^2 + v_b^2 \right) \frac{\delta}{\delta} \left( \chi_b - \chi_{b+} \right) - \frac{\left( \chi'w' \right)_b}{\delta} - \chi_b,
\]

where \( u_b \) and \( v_b \) are the radial and azimuthal components of wind speed in the boundary layer, \( \chi_b \) is a scalar quantity, taken here to be the dry static energy or the specific humidity, \( v_{gr}(r) \) is the tangential wind speed and \( w_b \) the vertical velocity at the top of the boundary layer, \( w_{b-} = \frac{1}{2}(w_b - |w_b|) \), \( f \) is the Coriolis parameter, \( C_D \) is the surface drag coefficient, \( \chi_b \) is the surface transfer coefficient for \( \chi_b \), \( \chi_{b+} \) is the value of \( \chi \) just above the boundary layer, \( \chi_s \) is the value of \( \chi \) at the sea surface, and the terms \( \left( u'w' \right)_b, v'w' \) represent turbulent fluxes at the top of the boundary layer. In the case of temperature, \( \chi_s \) is the sea surface temperature and in the case of moisture it is the saturation specific humidity at this temperature. The terms \( -c_p T_b \) and \( C_D (u_b^2 + v_b^2)^2 \) are added to Eq. (3) when \( \chi \) is the dry static energy and represent the effects of radiative cooling and the generation of enthalpy by frictional dissipation, respectively (here \( c_p \) is the specific heat of air at constant pressure). Bister and Emanuel (1998) showed that the dissipation term is significant at hurricane-strength wind speeds. We do not allow for condensation with latent heat release in the boundary layer and check that the boundary layer does not saturate (although the cloud base will become lower as the boundary layer humidity increases). The quantities \( u_b, v_b \), and \( \chi_b \) are assumed to be independent of depth. Note that \( w_{b-} \) is nonzero only when \( w_b < 0 \), in which case it is equal to \( w_b \). Thus the terms involving \( w_{b-} \) represent the transport of properties from above the boundary layer that may be different from those inside the boundary layer. A derivation of the equations is given in S03, although in that paper the flux terms were not explicitly included in the derivation, but were added later.

S03 evaluates \( C_D \) from the formula \( C_D = C_{D0} + C_{D1} |u_b| \), where \( C_{D0} = 1.1 \times 10^{-3} \), \( C_{D1} = 4 \times 10^{-3} \) and \( u_b = (u_b, v_b, 0) \). He assumes also that \( C_D = C_D \). We use this formulation for the comparison with S03’s results in Section 4. Recently Black et al. (2007) presented new aircraft measurements of the exchange coefficients for wind speeds up to 30 m s\(^{-1}\). These measurements suggest that \( C_D \) no longer increases for wind speeds higher than about 20 m s\(^{-1}\), although there is considerable scatter in the data. Therefore, for the calculations in later sections, we take \( C_{D0} = 0.7 \times 10^{-3} \) and \( C_{D1} = 6.5 \times 10^{-5} \) for wind speeds less than 20 m s\(^{-1}\) and \( C_D = 2.0 \times 10^{-3} \), a constant, for larger wind speeds. These values are based on our interpretation of Black et al.’s Fig. 5. For \( \chi \), we simply take a constant value equal to \( 1.1 \times 10^{-3} \), based on their Fig. 6. We found that the solutions are relatively insensitive to these differences in the exchange coefficients.

Substitution of Eq. (4) into Eq. (1) gives an expression for \( w_{b-} \):

\[
w_{b-} = \frac{\delta}{1+\alpha} \left[ \frac{1}{u_b} \left( \frac{v_{gr}^2 - v_b^2}{r} + f(v_{gr} - v_b) \right) + \frac{C_D}{\delta} \left( u_b^2 + v_b^2 \right) \frac{\delta}{\delta} u_b - \frac{u_b}{r} \right],
\]
where $\alpha$ is zero if the expression in square brackets is negative and unity if it is positive. With this expression for $u_\delta$, Eqs. (2) - (4) form a system that may be integrated radially inwards from some large radius $R$ to find $u_b$, $v_b$, and $\chi_b$ as functions of $r$, given values of these quantities at $r = R$. When $\chi_b$ represents the specific humidity, the sea surface temperature, $T_s$, and surface pressure, $p_s$, are required to determine the saturation specific humidity at the surface needed to calculate the surface moisture flux. While $T_s$ is taken to be constant, $p_s$ is calculated simultaneously with the boundary-layer quantities by integrating the gradient wind equation in the form $dp_s/dr = \rho v_{gr}^2/r + f v_{gr}$. \\

2.2 Representation of shallow convection

An important feature of the convective boundary layer over the tropical oceans in regions of large-scale subsidence is the near ubiquity of shallow convection. Such regions include the outer region of hurricanes. Shallow convection plays an important role in the exchange of heat and moisture between the subcloud layer, the layer which is modelled in this paper, and the cloudy layer above. As we do not predict the thermodynamic variables represented by $\chi_b$, above the boundary layer, we simply choose a constant value for the mass flux of shallow convection, $w_{sc}$, and add this to $w_{\delta}$ in Eqs. (1) - (3) (even if $w_{\delta} = 0$). This is equivalent to representing the flux terms by $w_{\delta}^\prime$ in these equations by $w_{sc}(\eta_+ - \eta_b)$, where $\eta$ is one of the dependent variables $u$, $v$, $\chi$, and a subscript ‘+’ denotes a value just above the boundary layer. However, $w_{\delta}$ in Eq. (4) is left unchanged as shallow convection does not cause a net exchange of mass between the cloud and subcloud layers. The value for $w_{sc}$ is chosen to ensure that the thermodynamic profile at $r = R$ is in radiative-convective equilibrium as explained in the next subsection.

2.3 Starting conditions at large radius

We assume that at $r = R$, far from the axis of rotation, the flow above the boundary layer is steady and in geostrophic balance with tangential wind speed $v_{gr}(R)$. In addition we take $C_D$ to be equal to $C_{D0} + C_{D1}v_{gr}(R)$. Then $u_b$ and $v_b$ satisfy the equations:

$$f(v_{gr} - v_b) = \frac{w_{\delta} + w_{sc}}{\delta} u_b - \frac{C_D}{\delta} (u_b^2 + v_b^2)^{1/2} u_b,$$

and

$$f u_b = \frac{w_{\delta} + w_{sc}}{\delta} (v_b - v_{gr}) - \frac{C_D}{\delta} (u_b^2 + v_b^2)^{1/2} v_b.$$  

A first approximation to the solution may be obtained analytically after setting the first two terms on the right-hand-side of each equation to zero, i.e. by neglecting $\alpha$ and $\chi_b$.

An alternative assumption would be to assume a linearized form of the full equations, replacing the left-hand-sides of Eqs. 6 and 7 by

$$\xi(v_{gr} - v_b)$$

and $\zeta_0$, respectively, where

$$\xi = \frac{2v_{gr}}{r} + f$$

and

$$\zeta_0 = \frac{dv_{gr}}{dr} + \frac{v_{gr}}{r} + f.$$  

This approximate solution is used as a first guess in an iteration procedure for $w_{\delta}$ that ensures zero moisture tendency at $r = R$. Zero moisture tendency requires that the rate of moisture gain from the sea surface is balanced by the loss of moist air through the top of the boundary layer and its replacement by dry air. The balance is expressed by the equation

$$C_\chi(u_b^2 + v_b^2)^2(q_s - q_b) = w_{sc}(q_b - q_{\delta_b}),$$

which, given the other quantities, is an equation for $w_{sc}$. When $w_{\delta}(R)$ and $w_{sc}$ have been determined, Eqs. (6) and (7) are solved again to find new values of $u_b$ and $v_b$ and the whole procedure is repeated until stable values are obtained for the four quantities. The iteration requires the sea surface temperature together with the specific humidities in the boundary layer, $q_b$, and just above the boundary layer, $q_{\delta_b}$, to be specified at $r = R$.

With the final value for $w_{sc}$ obtained, we calculate the temperature just above the boundary layer, $T_{\delta_b}$, so that for a specified radiative cooling rate and air temperature just above the surface, $T_{as}$, the sensible heat fluxes are in equilibrium at $r = R$. Then

$$T_{\delta_b} - T_{\delta} = \frac{1}{w_{sc}} \left[ \frac{R_b \delta}{c_p} - C_\chi(u_b^2 + v_b^2)^{1/2}(T_s - T_{as}) - \frac{C_D}{c_p} (u_b^2 + v_b^2)^{3/2} \right]$$

where $T_{\delta}$ is the temperature just below the boundary layer top. The temperature structure in the boundary layer including both $T_b$ and $T_{\delta}$ is determined on the assumption that the dry static energy is uniform across the boundary layer. The last term in square brackets is the dissipative heating. This is included for completeness although at $r = R$ it is small compared with other terms.

3 Comparison with S03

We examine first a calculation with the same parameters as in the control calculation described in section 6 of S03, considering here only the dynamical fields. The boundary layer depth is 550 m and $w_{sc}$ is $-2.2$ cm s$^{-1}$. Figure 1 shows a comparison of the radial and tangential wind components and the total wind speed in the boundary layer in the control calculation (panel (a)) and that in S03 (panel (b)). It shows also the tangential wind speed at the top of the boundary layer. In this case, there are significant quantitative differences in the corrected calculations compared with that in S03. The tangential wind speed in the boundary layer is mostly lower beyond the radius of maximum tangential wind speed above the boundary layer, $r_m$, but increases steeply as $r$ approaches...
inside a radius of 41.5 km it is supergradient and exceeds the maximum \( v_{gr} \) by 8 m s\(^{-1}\) at the radius where the solution breaks down.

The radial wind speed is about twice as large as that in S03 and reaches its maximum at about 50 km (1.25\( r_m \)), compared to a little more than 80 km (2\( r_m \)) in S03. However, \( u_b \) becomes zero at a radius of about 28.4 km (0.71\( r_m \)) at which point the boundary layer equations are singular. Near this radius, radial gradients are so steep that the underlying approximations of boundary-layer theory become questionable. The rapid decline in \( u_b \) near the singular radius implies a large vertical velocity at the top of the boundary layer. Indeed, the maximum upflow is much larger than that in S03, exceeding several m s\(^{-1}\) near the radius where the solution breaks down. In S03 it is only 0.15 m s\(^{-1}\) and occurs 1 km inside \( r_m \).

Since the results of the new calculation exhibit a behavior that was not found in earlier studies (e.g. Smith 1968, Leslie and Smith 1970, Bode and Smith 1975) as well as in S03, we began a thorough investigation of the dynamical and thermodynamical aspects of the boundary layer, including checks using two independent codes (one a Fortran90 code and the other using Mathematica). These studies led to what we believe are significant new findings, which are discussed below.

4 The new calculations - dynamical aspects

4.1 Dependence on boundary-layer depth

S03 investigated only a single boundary-layer depth, which was chosen to be that of the subcloud layer in a very simple model for radiative-convective equilibrium at the starting radius. However it is of some theoretical interest to enquire how the boundary-layer depth might influence the inward evolution of the layer since it is clear from Eqs. (1)-(3) that the "effective" enthalpy and moisture exchange coefficients are inversely proportional to the assumed depth (e.g. the effective frictional stress is the surface stress divided by the boundary layer depth).

For the remaining calculations we incorporate additional modifications to the model described by S03. Specifically we use the most recent representations of the drag and heat/moisture exchange coefficients based on the observations reported by Black et al. (2007) and we implement the new convective equilibrium scheme described in section 2c. These changes required the choice of slightly different values for the thermodynamic input parameters to achieve an equilibrium state. These parameters are: \( P_s = 1015 \) mb, \( T_s = 29^\circ C \), \( T_{as} = 28.5^\circ C \), \( q_b = 14 \) g kg\(^{-1}\), \( q_{b+} = 13.4 \) g kg\(^{-1}\), and \( R_b = 2.4^\circ C \) day\(^{-1}\). These lead to values for \( T_{b+} \) and \( w_{sec} \) of 21.7\(^\circ C \) and \(-5.7 \) cm s\(^{-1}\), respectively.

The results of calculations similar to those described in section 3, but for boundary layer depths 550 m and 800 m are summarized in Fig. 2, which shows graphs similar to those in Fig. 1. The flow behaviour in the calculation for \( \delta = 550 \) m (Fig. 1a) is similar to that for \( \delta = 550 \) m in S03 (Fig. 1b). However, the solution becomes singular (i.e. \( u_b \rightarrow 0 \)) at a larger radius: 35 km compared with 28.4 km. In addition, the maximum radial wind speed is lower (16 m s\(^{-1}\) compared with 21 m s\(^{-1}\)) and occurs at a slightly larger radius (54.7 km compared with 50 km). The maximum vertical velocity out of the boundary layer is less also: 1.8 m s\(^{-1}\) at \( r = 35 \) km compared with 3.8 m s\(^{-1}\) at \( r = 28.4 \) km in the case with \( \delta = 550 \) m. As the boundary layer depth increases to 679 m, the radius at which the solution becomes singular increases to 40 km, the maximum radial wind speed decreases to 14 m s\(^{-1}\), and the radius at which it occurs increases to 63 km. The maximum vertical velocity out of the boundary layer is slightly smaller, 1.6 m s\(^{-1}\), and occurs at \( r = 40 \) km.

As \( \delta \) increases beyond 679 m, a dramatic transition occurs in the solution behaviour, exemplified by the solution for 800 m (Fig. 2b). For \( \delta = 680 \) m and beyond, the solution for \( r < r_m \) is quite different from that for

\[ r_m \text{ is the radius at which the solution breaks down.} \]
Figure 2. Comparison of radial profiles of radial ($u_b$) and tangential ($v_b$) wind components and the total wind speed $vv$ in the boundary layer as well as the tangential wind speed above the boundary layer ($v_{gr}$), but for four calculations with boundary layer depths (a) 550 m, and (b) 800 m. [Units m s$^{-1}$]

Figure 3. Radial profiles of vertical velocity ($w_\delta$) at the top of the boundary layer in the calculations with boundary layer depths of 550 m and 800 m. [Units cm s$^{-1}$]

$\delta \leq 679$ m and extends to within a few kilometres of the rotation axis. In this "large depth" regime, the tangential wind speed in the boundary layer becomes subgradient again after reaching its peak supergradient value. Thereafter it oscillates about the prescribed wind profile above the boundary layer with ever decreasing amplitude as the axis is approached. The oscillations are accompanied by oscillations of the radial wind field and therefore in the vertical flow at the top of the boundary layer. This behaviour is similar to that described in S03 for a vortex with $r_m = 100$ km.

The vertical motion at the top of the boundary layer in the calculations with boundary layer depths of 550 m and 800 m are shown in Fig. 3. There is a slight adjustment near the starting radius on account of the sudden introduction of the inertial acceleration terms in the boundary layer, but the subsidence velocities at outer radii are relatively weak. The subsidence increases with decreasing radius and then decreases again shortly before changing to ascent. The change from subsidence to ascent occurs at a radius of 130 km when $\delta = 550$ km and 155 km when $\delta = 800$ km. Reasons for these differences are discussed below.

The dependence of the solutions on the boundary-layer depth is succinctly summarized by plots of the maximum radial and tangential components of wind speed and the radii at which these occur as functions of $\delta$ (Fig. 4). As $\delta$ increases, the effective frictional stress (i.e. the surface stress divided by the boundary-layer depth) decreases, and the degree of supergradient flow progressively diminishes. This behaviour is shown by the difference between the maximum tangential wind speed in the boundary layer, $v_{bmax}$, and the tangential wind speed above the boundary layer at the radius $r_v$ at which $v_{bmax}$ occurs (Fig. 4a). In contrast, $r_v$ increases with increasing $\delta$ (Fig. 4b). The maximum inflow, $u_{bmax}$, decreases also
4.2 Interpretation

The foregoing behaviour depends in a delicate way on the relative importance of various force terms in the radial and tangential components of the momentum equation. To aid the interpretation we rewrite Eqs. 1 and 2 in the following form:

\[
\frac{du_s}{ds} = \frac{u_s}{\delta} - \frac{(v_{gr} - v_b)}{u_s} \left( \frac{v_{gr} + v_b}{R - s} + f \right)
= -\frac{C_D}{\delta} (u_s^2 + v_b^2) \left( \frac{u_s}{u_s} + \frac{v_b}{R - s} + f \right),
\]

(11)

\[
\frac{dv_b}{ds} = \frac{w_{bc} + w_{sc}}{\delta} \frac{v_b - v_{gr}}{u_s} + \frac{v_b}{R - s} + f
= -\frac{C_D}{\delta} (u_s^2 + v_b^2) \left( \frac{v_b}{u_s} \right),
\]

(12)

where \( u_s = -u_x \) is the radial inflow velocity and \( s = R - r \), with \( s \leq R \), measures distance inwards from the starting radius, \( R \). In addition we have replaced the flux terms on the far right of Eqs. (1) and (2) with the formulation described in section 2.2. In this form the equations show how the (inward) radial and tangential components of flow change with decreasing radius.

In the absence of frictional stresses, converging rings of air would conserve their absolute angular momentum, \( rv + \frac{1}{2} f r^2 \), and spin faster (here \( v \) is the tangential component of wind speed). In the boundary layer, they still spin faster, but the rate at which \( v_b \) increases is reduced by the frictional torque. This effect is represented by the last term in Eq. (12). The development of supergradient winds requires a sufficiently large radial displacement of air parcels in the boundary layer, which in turn requires sufficiently-large, radially-inward wind speeds. From a Lagrangian viewpoint one may think of air parcels spiralling inwards: the slower they move inwards, the longer tracks they have along which friction can act to reduce \( v_b \). This effect is contained in the terms proportional to the inverse of \( u_s \) in Eq. (12). It is clear from the foregoing discussion that the development of supergradient winds depends on the radial gradient of absolute angular momentum in the boundary layer and hence on that above the layer, a feature explored in the context of a linear boundary-layer model by Kepert (2001) and Kepert and Wang (2001).

Equation (11) shows that the only term that can cause a radially-inward acceleration in the slab model is the net pressure gradient, the effect of which is contained in the second term on the right-hand-side. The first and third terms in this equation represent the effects of the downward transport of radial momentum (zero in the present model), and the frictional stress, respectively, and both of these act to reduce the radial inflow. If the flow is supergradient, i.e. if \( v_b > v_{gr} \), the net pressure gradient acts radially outwards also.

The net inward pressure gradient increases with the degree to which the tangential flow in the boundary layer is subgradient (i.e. to \( v_{gr} - v_b \)), which in turn increases as the effective frictional torque increases. Equation (12) shows that this torque is the only term that leads to a reduction of \( v_b \) with decreasing radius as long as the flow in the boundary layer remains supergradient. Since the friction terms are inversely proportional to the boundary-layer depth, it follows that shallower boundary layers...
favour lower tangential wind speeds, but larger radial wind speeds, because, at least in the outer part of the vortex, they lead to a larger net pressure gradient. At inner radii the situation is a little different. Then the term $v_b/(R-s)$ in Eq. (12) becomes large and contributes to an increase in $v_b$ with $s$. Thus larger radial wind speeds favour larger tangential wind speeds because then air parcels may penetrate rapidly to smaller radii where this effect is important. In addition they suffer less total frictional torque on the way (note that the frictional term in Eq. (12) decreases as $u_s$ increases).

The key to what determines the two flow regimes depends on which of the foregoing processes dominates and boils down to whether or not the flow can become subgradient again before $u_b$ becomes zero. In the calculation with $\delta = 680 \text{ m}$, the tangential flow just manages to become subgradient before $u_b$ becomes zero, whereupon the inflow begins to increase again with decreasing radius. Because the tangential wind speed at the top of the boundary layer decreases also, the flow again becomes supergradient so that $u_b$ decreases rapidly and $v_r - v_{gr}$ decreases until $u_b$ becomes subgradient again. These fluctuations are of supergradient wind speed is a maximum for an intermediate value of $\delta$ in the range shown. The reason for the foregoing behaviour is that the downward mixing of radial momentum by shallow convection reduces the strength of the inflow directly, while that of azimuthal momentum reduces the net inward pressure gradient, which reduces the inflow indirectly. The reduced inflow diminishes the strength of supergradient winds that can be achieved. For a particular value of $w_{sc}$, these effects are reduced by a decrease in $\delta$, which increases the effective frictional force in the boundary layer. The radii at which the maxima occur (panel (b)) as functions of $\delta$ for the three values of $w_{sc}$. It is seen that for a fixed value of $\delta$, the maximum of both the radial and tangential components increases as $|w_{sc}|$ increases. The maximum in the inflow increases also with decreasing boundary-layer depth, and this is true of the degree of supergradient wind speed for $w_{sc} = 0$. However for $|w_{sc}| = 5$ and 10 cm s$^{-1}$, the degree of supergradient wind speed is a maximum for an intermediate value of $\delta$ in the range shown. Nevertheless it is pertinent to ask how sensitive the foregoing results are to the magnitude chosen for $w_{sc}$. To answer this question we carried out two additional calculations similar to the control calculation, but with $w_{sc} = 0$ in one and $w_{sc} = -10 \text{ cm s}^{-1}$ in the other. It turns out that for $w_{sc} = 0$, the transition in boundary layer behaviour described in section 4.1(a) occurs at a larger boundary layer depth (765 m instead of 680 m). In the case where $w_{sc} = -10 \text{ cm s}^{-1}$, there is no transition in behaviour for any boundary layer depth. The tangential wind speed in the boundary layer becomes subgradient again after reaching its maximum value and then oscillates about the prescribed wind profile for any boundary layer depth. At the same time the radial wind and vertical flow oscillate. The complete range of effects is summarized in Fig. 5, which plots the maxima of radial and tangential wind speed in the boundary layer (panel (a)) and the radii where the maxima occur (panel (b)) as functions of $\delta$ for the three values of $w_{sc}$. The dynamical interpretations given above provide also an explanation for the differences in the radial location where $w_{sc}$ changes sign in Fig. 3. The larger effective friction for the shallower boundary layer implies a larger net radial pressure gradient, which, in turn, leads to a larger acceleration of the radial flow and a decrease in the radius at which the radial gradient of inward mass flux changes sign.

4.3 Dependence on vortex intensity

Decreasing the vortex intensity has a similar effect to increasing the boundary layer depth. A repeat of the control calculation for different values of the maximum tangential wind speed at the top of the boundary-layer, $v_m$, shows that as $v_m$ decreases, the strength of supergradient winds decreases. In addition, the transition in regime from one in which $u_b$ becomes zero before $v_b$ reduces to $v_{gr}$ to one in which $v_b$ oscillates about $v_{gr}$ occurs at a smaller boundary-layer depth. For example, when $\delta = 550 \text{ m}$, the regime transition occurs if $v_m$ is reduced by just 8 m s$^{-1}$ to 32 m s$^{-1}$. As $v_m$ decreases further, the behaviour is similar to that when $\delta$ decreases at fixed $v_m$. These findings are consistent with the results of Kepert (2001). In his Figure 1 he shows that a larger gradient wind speed leads to a stronger jet, i.e. to an increase in the strength of supergradient winds. The behaviour discussed above suggests that it might be possible to re-scale the equations in a way that the $v_m$ and $\delta$ dependence condenses into a single parameter, but this does not appear to be the case.

4.4 Dependence on mixing by shallow convection

S03 showed that it is important to include a representation of downward mixing by shallow convection to prevent the boundary layer from completely saturating. The formulation is necessarily crude because thermodynamic quantities are not predicted above the boundary layer. This means that there is no physical basis for allowing the mass transport due to shallow convection to vary with radius. Nevertheless it is pertinent to ask how sensitive the foregoing results are to the magnitude chosen for $w_{sc}$. To answer this question we carried out two additional calculations similar to the control calculation, but with $w_{sc} = 0$ in one and $w_{sc} = -10 \text{ cm s}^{-1}$ in the other. It turns out that for $w_{sc} = 0$, the transition in boundary layer behaviour described in section 4.1(a) occurs at a larger boundary layer depth (765 m instead of 680 m). In the case where $w_{sc} = -10 \text{ cm s}^{-1}$, there is no transition in behaviour for any boundary layer depth. The tangential wind speed in the boundary layer becomes subgradient again after reaching its maximum value and then oscillates about the prescribed wind profile for any boundary layer depth. At the same time the radial wind and vertical flow oscillate. The complete range of effects is summarized in Fig. 5, which plots the maxima of radial and tangential wind speed in the boundary layer (panel (a)) and the radii where the maxima occur (panel (b)) as functions of $\delta$ for the three values of $w_{sc}$. It is seen that for a fixed value of $\delta$, the maximum of both the radial and tangential components increases as $|w_{sc}|$ increases. The maximum in the inflow increases also with decreasing boundary-layer depth, and this is true of the degree of supergradient wind speed for $w_{sc} = 0$. However for $|w_{sc}| = 5$ and 10 cm s$^{-1}$, the degree of supergradient wind speed is a maximum for an intermediate value of $\delta$ in the range shown. The reason for the foregoing behaviour is that the downward mixing of radial momentum by shallow convection reduces the strength of the inflow directly, while that of azimuthal momentum reduces the net inward pressure gradient, which reduces the inflow indirectly. The reduced inflow diminishes the strength of supergradient winds that can be achieved. For a particular value of $w_{sc}$, these effects are reduced by a decrease in $\delta$, which increases the effective frictional force in the boundary layer. The radii at which the maxima in $u_b$ and $v_b$ occur increase with $\delta$, the rate-of-increase being largest for the case with no mixing.

The maximum amount by which the tangential wind becomes supergradient (graphs not shown) is sensitive to changes in $w_{sc}$ and decrease significantly as $|w_{sc}|$ increases. The maximum vertical flow at the top of the boundary layer decreases a little also and the radius at which it occurs increases.
4.5 Effects of radially-varying boundary-layer depth

The model described in section 2 assumes a constant boundary layer depth, although a scale analysis of the equations as well as the linear solution to the full boundary-layer equations (Eliassen and Lystad 1977, Kepert 2001) suggests that the depth should decrease with declining radius at a rate \( \sqrt{f/I} \), where \( I = (f + \zeta)(2v_{gr}/r + f) \) and \( \zeta = (1/r)d(rv_{gr})/dr \) is the vertical component of relative vorticity at the top of the boundary layer. While it is not possible to determine the radial variation of \( \delta \) in the slab-model, it is straightforward to modify the Eqs. (1) - (4) to allow for a prescribed variation \( \delta(r) \) (see Appendix). To assess the effect of a decrease in \( \delta \) with declining radius we carried out a calculation in which \( \delta(r) = \delta(R)\sqrt{(f/I)} \), where \( \delta(R) \) is the boundary layer depth at \( r = R \). The solutions for \( \delta(R) = 550 \) m and 800 m are shown in Fig. 7. In both cases the tangential wind speeds in the boundary layer are decreased, especially inside a region of about 200 km and the peak winds are significantly lower in magnitude than \( v_m \). In contrast the peak radial winds are larger than in the constant-depth calculations, especially in the calculation for \( \delta(R) = 800 \) m and the maxima occur at markedly smaller radii. These differences in behaviour are consistent with the ideas presented in section 4.2, noting that a decreasing boundary-layer depth implies a larger effective drag through the layer. When the boundary-layer depth decreases with decreasing radius, the maximum vertical velocity at the top of the layer is reduced considerably from that in the constant-depth calculations and is more in line with that in previous calculations (e.g. Kepert and Wang 2001: see e.g. their Fig. 3). The reducing-depth calculations still show slightly supergradient wind speeds and oscillations in radial and vertical motion, but now well inside \( r_m \) and again in a region where radial gradients are probably steep enough to invalidate the assumptions of boundary layer theory.

4.6 Effects of downward momentum transport

The calculations of S03 showed that where there is inflow into the boundary layer \( (u_g < 0) \), the contribution of the terms involving \( w_{\delta-} < 0 \) to the radial derivatives on the left of Eqs. (1) - (3) is small. This suggests that a simplified approximate system of equations could be obtained by setting \( w_{\delta-} = 0 \) in these equations and by diagnosing \( u_g = 0 \) using the continuity equation, Eq. (4). We explore here the accuracy of this approximation in the case where the boundary-layer depth is a constant.

Figure 8 compares the radial and tangential wind components in the boundary layer and the vertical motion at the top of the boundary layer in two calculations, one like the control calculation, but with a boundary layer...

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Figure 5. (a) Maximum radial and tangential wind speeds \((u_n, v_n)\) respectively, \( n = 1 \rightarrow 3 \) in the boundary layer and the tangential wind speed at the top of the boundary layer at the radius where the maximum \( v_n \) occurs as functions of boundary-layer depth for three different values for \( w_{sc} \): 0 cm s\(^{-1}\); 5 cm s\(^{-1}\); and 10 cm s\(^{-1}\). (b) Radii at which the maximum radial and tangential wind speeds occur \((r_{u_n} \text{ and } r_{v_n} \text{, respectively})\) as functions of boundary-layer depth.

Figure 6. Assumed radial variation of boundary-layer depth, \( \delta(r) \), for the calculation shown in 7.
4.7 Vertical motion at the top of the boundary layer

The formula for the vertical velocity at the top of the boundary layer (Eq. 5) differs considerably from that derived by Kepert (2001, Eq. 28) based on a linear approximation to the full boundary-layer equations and, indeed, from that obtained from a linear approximation to the slab model (see appendix B). The differences in vertical motion predicted by these different formulae for the tangential wind profile $v_{gr}$ used herein are shown in Fig. 9. It is seen that the vertical velocity profile in our full nonlinear model is more peaked than in the approximate theories and the maximum upflow velocity is more than twice that of the linear theories and occurs at a significantly smaller radius. The linear slab model and the more complete version give similar profiles and similar maxima, but the maximum inflow occurs at a larger radius in the slab model.
5 The new calculations - thermodynamical aspects

5.1 Dependence on boundary-layer depth

The solutions for the thermodynamic variables obtained in the new calculations are qualitatively similar to those obtained by S03, but there are some quantitative differences. Since the present calculations include, inter alia, an improved algorithm for calculating the radiative-convective equilibrium state at the starting radius and have slightly different parameter values, a detailed quantitative comparison is not warranted. We show two of the new solutions here for constant boundary layer depths of 550 m and 800 m. The details are summarized in Fig. 10. Panels (a) and (c) of this show the radial profiles of the boundary layer temperature, specific humidity, saturation specific humidity, and saturation specific humidity at the sea surface, \( q_{ss} \) for the two solutions, while panels (b) and (d) of Fig. 10 show the sensible and latent heat fluxes at the surface and through the top of the boundary layer.

The boundary layer temperature is nearly constant in both cases with a value of about 25.8 °C for δ = 550 m and 24.5°C for δ = 800 m, but shows a small rise in the inner core region at radii less than about 100 km. In essence, the mean boundary layer temperature largely follows the sea surface temperature, but since the temperature in the boundary layer decreases adiabatically with height, \( T_b < T_{SST} \). The increase in the core region is associated with dissipative heating, which appears to be significant at high wind speeds. Consistent with this heating, the sensible heat fluxes are slightly negative in the core region. Recently Smith (2006, 2007) showed that an inviscid balanced vortex where the tangential circulation decays with height is cold cored at the surface. The present calculations show that this is not the case when one accounts for the boundary-layer effects, neglecting, of course, ocean surface cooling brought about by upwelling induced by the strong surface winds and the effects of unsaturated convective downdrafts (Cione et al. 2000). The results suggest also that the effects of adiabatic cooling as air parcels move inwards towards lower pressure is more than outweighed by the sensible heat fluxes and, in the core region, by dissipative heating.

The specific humidity increases markedly with decreasing radius from a value of 14.5 g kg\(^{-1}\) at \( r = 500 \) km to about 17.8 g kg\(^{-1}\) at \( r = 35 \) km. This increase is associated with an increasing surface moisture flux, which outweighs the flux of dry air through the top of the boundary layer (panels (b) and (d)). The saturation specific humidity, \( q_{ss} \), for a boundary layer depth of 550 m varies between 21.3 g kg\(^{-1}\) and 23.2 g kg\(^{-1}\) for radii between 500 km and about 35 km. The values for the deeper boundary layer (δ = 800 m) are typically about 1.5 g kg\(^{-1}\) smaller. It is interesting to note that in both cases, \( q_b < q_{ss} \) at all radii. Thus the air does not become saturated near the sea surface, but the lifting condensation level lowers as the boundary layer moistens.
Figure 10. Radial profiles of boundary-layer temperature, $T_b$ (in °C), specific humidity, $q_b$, saturation specific humidity, $q_{sb}$, and the saturation specific humidity at the sea surface, $q_{ss}$, all in g kg$^{-1}$, for boundary-layer depth (a) 550 m and (b) 800 m. Panels (c) and (d) show latent and sensible heat fluxes from the sea surface, ($\text{flux}_q$ and $\text{flux}_h$, respectively) and through the top of the boundary layer ($\text{flux}_{qt}$ and $\text{flux}_{ht}$) the two boundary-layer depths. All fluxes are given in W m$^{-2}$. The sign of $\text{flux}_{ht}$ has been reversed for convenience of plotting.

The latent heat fluxes are much larger than the sensible heat fluxes and they increase strongly with decreasing radius. The increase of the surface flux reflects the fact that this flux increases with the near-surface wind speed and with the degree of disequilibrium between specific humidity of the air near the surface and the saturation specific humidity at the sea-surface temperature. Since the latter increases with decreasing pressure, the degree of disequilibrium is maintained (see Figs. 10a and 10c) and, of course, the wind speed increases with decreasing radius to a radius of 40 km. The increase in boundary layer moisture increases the moisture contrast at the top of the boundary layer, since the specific humidity of air above the boundary layer is held constant in the present model. It is this increase in moisture contrast that accounts for the increase in magnitude of the latent heat flux at the top of the boundary layer as the radius decreases. This increase in moisture contrast is not likely to be realistic as convection would be expected to progressively increase the moisture content of the air above the boundary layer with decreasing radius. For this reason, we would expect the predicted increase in $q_b$ with decreasing radius to be a lower bound of that which would occur in reality.

The curves for the latent heat flux at the top of the boundary layer show a kink at the radius where $w_\delta$ changes sign (about 130 km for $\delta = 550$ m and 150 km for $\delta = 800$ m). Inside these radii, $w_\delta$ is zero and does not contribute to the fluxes at $z = \delta$. At very large radii, $w_{sc}$ dominates so that the moisture flux terms are similar for both values of $\delta$, but as the radius decreases, $w_\delta$ becomes significant and the terms diverge from one another.

There is less sensitivity of the thermodynamic quantities to a radially-varying boundary-layer depth compared with dynamical quantities. As expected, calculations for $\delta(R) = 550$ m and $\delta(R) = 800$ m give values of temperature and humidity at large radii similar to those in the
case with constant $\delta$ (Figure not shown), but the values at the radius of maximum tangential wind speed above the boundary layer are slightly higher (about 1°C in the case of $T_e$ and up to 1.5 g kg$^{-1}$ in the case of $q_b$) than those shown in Fig. 10. In the varying-depth calculations, $T_e$ and $q_b$ reach nearly the same peak values ($T_e = 28.5$ C and $q_b = 15$ g kg$^{-1}$) for both values of $\delta(R)$ because the boundary layer depths become similar at inner radii. Because vertical velocities in these calculations are much less than in the constant depth ones, there are no discernible kinks in the moisture fluxes at the top of the boundary layer like those in Fig. 10.

A thermodynamic quantity of fundamental theoretical interest is the reversible equivalent potential temperature, $\theta_e$, as it has been used in developing a theory for the potential intensity of tropical cyclones (Emanuel 1986, 1988, 1995, Bister and Emanuel 1998). For this reason we show in Fig. 11 the radial variation of $\theta_e$ for the case with a boundary layer depth that varies as described in section 4.5. In this figure, $\theta_{e1}$ and $\theta_{e2}$ label the curves for $\delta(R) = 550$ m, and $\delta(R) = 800$ m, respectively. In both calculations, $\theta_e$ increases with decreasing radius, while a deeper boundary layer leads to marginally lower values.

Recently Montgomery et al. (2006) and Bell and Montgomery (2007) presented observational data from the category five Hurricane Isabel 2003 including data showing the radial increase of $\theta_e$ towards the centre. Such an increase is shown by our model also. To be able to compare the predictions of our model quantitatively with their measurements we carried out two more calculations for the same boundary-layer depths, but with a maximum tangential wind speed of 70 m s$^{-1}$, which is more appropriate for a category five storm like Isabel. The two $\theta_e$-curves for these calculations are shown also in Fig. 11 where they are labelled $\theta_{e3}$ (for $\delta(R) = 550$ m) and $\theta_{e4}$ (for $\delta(R) = 800$ m). As expected, $\theta_e$ reaches higher values than for the weaker vortex, but the difference between the values of $\theta_e$ for the two values of $\delta$ is larger. For $\delta(R) = 550$ m and a maximum tangential wind speed of 70 m s$^{-1}$, the solution breaks down at a radius of about 60 km, but the solutions up to this point show a steady increase in $\theta_e$, which reaches a value of approximately 355 K. This value is not outrageous compared with the values reported by Montgomery et al. (2006), especially considering the crudity of holding $w_{sc}$ and $q_{bs+}$ constant with radius in our model. For example, on 12 September 2003 they found values of $\theta_e$ of about 353 K at radii between 50 and 60 km in the low-level inflow layer and up to 360 K on 12 September (see Fig. 5 in their paper). Moreover these values continued to rise with decreasing radius as they do in our model until the model breaks down. Montgomery et al. calculated $\theta_e$ pseudo-adiabatically, but at low-levels in the boundary layer where there is no liquid water, these should be essentially the same.

6 Limitations and Implications

Although many simple hurricane models represent the boundary layer as a single layer of fixed depth, such formulations have a number of limitations and these should be borne in mind when interpreting the results of our study. Some important limitations are listed below together with certain implications of the results.

- One assumption in deriving the bulk equations is that the vertical average of terms such as those representing radial advection are equal to the radial advection computed from vertically-averaged quantities. This assumption may be expected to be inaccurate if there are regions of strong outflow overlying regions of inflow as happens near the radius of maximum tangential wind speed in the continuous models (e.g. Kepert and Wang, 2001, Montgomery et al. 2001). However, this should be much less of an issue if the boundary layer is considered to be just the inflow layer, itself, as is the case here.

- The prescription of a uniform depth with radius is a further limitation as an elementary scale analysis suggests that the layer depth must decrease as the inertial stability increases, a fact that is confirmed by many solutions where the depth is allowed to vary (e.g. Smith 1968, Leslie and Smith 1970, Eliassen and Lystad 1977, Kepert 2001, Kepert and Wang 2001, Montgomery et al. 2001). However, this limitation may be removed as shown in section 4.5.

- At large radii, where there is mean subsidence into the boundary layer, the boundary-layer depth will be determined in reality by a subtle balance between the turbulence generated within it, which tends to deepen it, and the subsidence aloft, which tends to make it shallower. At inner radii, where there is
mean ascent out of the boundary layer, the vertical advection of turbulence may be expected to play a role also in determining the depth (see e.g. Stull, 1988, especially Fig. 1.6 and the related discussion). Such processes are clearly beyond the ability of a one-layer model to capture, but they have important implications for the veracity of the formulation at inner radii.

- The consequences of prescribing the tangential wind speed just above the boundary layer in the inner-core region, where the flow exits the boundary layer, are unclear. Many previous boundary-layer models have taken this approach (e.g. Smith 1968, Ooyama 1969, Leslie and Smith 1970, Bode and Smith 1975, Shapiro 1983, Kepert 2001, Kepert and Wang 2001, Smith 2003), but the consequences thereof have not been investigated or discussed in detail. In this region it would seem more reasonable to suppose that boundary-layer air carries its momentum with it as it ascends. In the slab model, this would imply that the exiting air carries the momentum $\rho(u_b, v_b)$ with it, but there is no reason to suppose that on exit, $v_b$ will be in gradient wind balance.

- The foregoing issue was tacitly recognized by Emanuel (1986) in his formulation of a steady-state hurricane model. In this model he assumed that the tangential flow just above the boundary layer ($v_{gr}$) is equal to that in the boundary layer ($v_b$) in regions of ascent and that it satisfies gradient wind balance. Emanuel makes a similar assumption also in his theory for potential intensity (see Bister and Emanuel 1998 and refs.). Of course, this assumption implies that gradient-wind balance exists in the boundary layer, which cannot be the case or there would be no net force to drive inflow!

- In the slab models, $(u_b, v_b)$ represents an average through the depth of the boundary layer so that some difference between $(u_b, v_b)$ and the imposed gradient flow $(0, v_{gr})$ at the top of the boundary layer could be tolerated. In these models, at least, one may regard the prescription of $v_{gr}$ as simply a specification of the radial pressure gradient and the only place in our formulation where this leads to inconsistency would be in the assumption that the momentum fluxes at the top of the boundary layer associated with shallow convection or precipitation-driven downdrafts are proportional to $(0, v_{gr}) - (u_b, v_b)$. Except for this assumption, the slab model would appear to be less constrained than those which have vertical structure and which assume not only that the flow exiting the boundary layer has a prescribed radial pressure gradient, but also that $(u(z), v(z)) \rightarrow (0, v_{gr})$ at the top of the boundary layer, where $v_{gr}$ is in gradient wind balance (here $z$ is the vertical coordinate).

- The mismatch between $(u_b, v_b)$ and $(0, v_{gr})$ predicted by the slab model suggests that the outflow jet found above the inflow layer in the full numerical solutions\(^5\) presented by Montgomery et al. (2001) is a means by which the flow exiting the boundary layer adjusts to the radial pressure gradient associated with the vortex above the boundary layer. The implication would be that a more complete formulation of the (steady) boundary layer in the inner core region of a tropical cyclone using a slab-type formulation would require at least two layers including one to represent the outflow jet. In this layer, the radial and tangential wind fields would need to adjust to the radial pressure gradient implied by the mass distribution in the free troposphere.

- We would argue that models which allow the boundary layer to have vertical structure do not avoid the problems of unrealistically over-constraining the flow that exits the boundary layer if they set $(u(z), v(z))$ equal to $(0, v_{gr})$ at the top of the layer. For example, the solutions reported by Kepert and Wang (2001, Fig. 2) show supergradient flow everywhere above the boundary layer (as defined by the region where there are significant turbulence levels) even in regions where turbulence levels are small and where there is no apparent radial or vertical motion. The reasons for these supergradient winds are hard to reconcile in terms of the insights gained from the slab model, which requires strong inflow to achieve supergradient winds. That is not to say that Kepert and Wang’s results are incorrect, but they need to be understood. It is significant that the numerical solutions of Montgomery et al. (2001), in which no constraint needs to be imposed at the top of the boundary layer, do not show any single level above the boundary layer where the radial flow is everywhere zero. We regard these issues as important ones requiring further research.

We would argue that the ubiquitous tendency of the slab model to produce supergradient winds is significant. A well-known result from the inviscid axisymmetric balanced theory of vortex intensification is that the latent heat release in eye-wall convection tends to produce a secondary circulation in which the tangential wind tendency is largest inside the radius of maximum tangential wind speed (Shapiro and Willoughby, 1982, p389) so that the vortex contracts as it intensifies. If the boundary layer tends to generate supergradient winds inside the radius of maximum tangential wind speed above it and if these winds are advected vertically out of the boundary layer, they would contribute in a similar way to a spin up of the core region. Such behaviour is consistent with calculations performed by our late colleague, Wolfgang Ulrich. Using an axisymmetric tropical-cyclone model, he found that the ring of air corresponding with the maximum calculated tangential wind speed always originated at large radial distances in the boundary layer. The idea is supported also by the simple tropical-cyclone model examined by

\(^5\) Note that in these solutions, the boundary layer and flow above are solved together.
Emanuel (1997) in which the inner-core spin up appears to be orchestrated by the boundary layer. The veracity of these results would indicate that the boundary layer is a fundamental aspect of the spin-up of the inner-core of a tropical cyclone, at least in the context of axisymmetric dynamics.

7 Conclusions

Corrected solutions to the simple model for the boundary layer of a hurricane by the first author show the development of supergradient winds to be a ubiquitous feature. The solutions exhibit two types of behaviour in the inner core of the vortex depending on the boundary layer depth and the maximum tangential wind speed above the layer. For small depths/large maximum tangential wind speeds, the winds are strongly supergradient and lead to a rapid deceleration of the inflow. As a result, the inflow becomes zero at some radius inside the radius of maximum tangential wind speed above the boundary layer, where the equations are singular. At this point and near it, boundary-layer theory is no longer applicable. At a particular depth, which increases with the maximum tangential wind speed above the boundary layer, there is an abrupt transition in behaviour and for depths larger than this value, the solutions remain nonsingular until within a few km of the rotation axis. Inside the radius of maximum tangential wind speed above the boundary layer, the tangential wind speed in the boundary layer oscillates about that above the layer, becoming alternately supergradient and subgradient. These oscillations are accompanied by oscillations in the radial wind speed in the layer and in the vertical flow at the top of the boundary layer. Interpretations were given for the behaviour found, but the significance of the oscillations in reality is unclear. They may be more an artifact of prescribing the radial pressure gradient, or equivalently the tangential wind speed, at the top of the boundary layer where there is upflow out of the boundary layer.

The results of other aspects of the boundary-layer dynamics and thermodynamics that were investigated are summarized briefly below.

- We showed that as the mass flux for shallow convection increases, the transition depth dividing the two boundary-layer regimes increases. This behaviour can be explained if one takes into account that the mixing of radial momentum by shallow convection reduces the inflow and with that the strength of supergradient winds in the boundary layer.
- We noted that increasing the drag coefficient has a similar effect on the solutions to decreasing the boundary layer depth δ. In general a higher drag coefficient leads to a larger value for the boundary layer depth at which the transition in solution behaviour occurs.
- Calculations in which the boundary layer depth is allowed to vary inversely with the square root of the inertial parameter give more realistic vertical velocities at the top of the boundary layer when compared with full numerical solutions, but thermodynamic quantities are less sensitive to the varying depth.

- We explored the accuracy of a simplified approximate system of the dynamical equations in which the mean vertical velocity at the top of the boundary layer is set equal to zero in the momentum equations and is simply diagnosed using the continuity equation. We found this approximation to be reasonably accurate.

- An improved algorithm was described for calculating the radiative-convective equilibrium state of the boundary layer at some large radius. Notwithstanding the inclusion of this scheme, as well as correcting the Runge-Kutta routine, the solutions for the thermodynamic variables obtained in the new calculations are qualitatively similar to those obtained by S03. We found that the radial variation of thermodynamical quantities had a relatively weak dependence on the boundary-layer depth.

- Predicted values of the equivalent potential temperature are in acceptable agreement with observations made in Category-5 Hurricane Isabel (2003) for a suitably intense vortex.

- Our results point to a potential inconsistency in hurricane boundary layer models that require the flow out of the boundary layer to have zero radial motion and a prescribed, balanced, tangential wind speed. This limitation applies to many previous studies of the boundary layer that we are aware of and is the subject of further study. For reasons we discussed, the restriction would appear to be less severe in slab models than in models that allow for vertical structure.

- The solutions for thermodynamic quantities suggest that heat and moisture fluxes at the top of the boundary layer are comparable in magnitude with those at the top of the boundary layer. While there have been recent attempts to improve measurements of the surface fluxes at high wind speeds, our calculations point to an urgent need for field measurements of the fluxes at the top of the boundary layer.

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Let the boundary layer depth, $\delta(r)$, be a specified function of radius $r$ and define

$$
\phi_0 = \frac{1}{\delta} \int_0^\delta \phi dz
$$

(13)

to be the vertical average of any quantity $\phi(r, z)$ across the boundary layer. Then

$$
\frac{d\phi_0}{dr} = \frac{1}{\delta} \int_0^\delta \frac{d\phi}{dz} dz - \frac{1}{\delta} \frac{d\delta}{dr} (\phi - \phi_0) ,
$$

(14)

where $\phi_0$ is the value of $\phi$ just above the boundary layer. Applying this equation to $w_0$, the continuity equation 

$$
\frac{1}{r} \frac{\partial r u}{\partial r} + \frac{\partial w_0}{\partial z} = 0 
$$

shows that $w_0$ must be evaluated using Eq. (15).

Equations (1) - (3) do not change in the variable-depth case, but the value of $w_0$ must be evaluated using Eq. (15). As a result, the expression (5) for $w_0$ contains the additional term $-u_0\delta r/dr$ on the right-hand-side, but it turns out that the contribution of this term is small.

9 Appendix B

The linearized form of the boundary-layer equations are

$$
- \left( \frac{2 v_{gr}}{r} + f \right) v_0 = - \frac{C_D}{\delta} (u_0^2 + v_0^2)^{1/2} u_0 ,
$$

(16)

$$
\left( \frac{dv_{gr}}{dr} + \frac{v_{gr}}{r} + f \right) u_0 = - \frac{C_D}{\delta} (u_0^2 + v_0^2)^{1/2} v_0
$$

(17)

where $v_0 = v_{gr} - v_{yr}$. Let $u = u_0/v_{gr}$ and $v = v_0/v_{gr}$. Then (16) and (17) become

$$
- \xi (v - 1) = - \frac{C_D v_{gr}}{\delta} (u^2 + v^2)^{1/2} u ,
$$

(18)

$$
\zeta u = - \frac{C_D v_{gr}}{\delta} (u^2 + v^2)^{1/2} v .
$$

(19)

Dividing (18) by (19) leads to the relation

$$
u^2 = \frac{\xi}{\zeta} v (1 - v) .
$$

(20)

Then squaring (18) and eliminating $u$ gives an algebraic equation for $v$, which after a little rearrangement may be written as

$$
(1 - v) = \mu^2 \delta^2 I^2 \left[ \frac{\xi}{\zeta} (1 - v) + v \right] v^2
$$

(21)

where

$$
I^2 = \zeta \xi , \quad \text{and} \quad \mu = \frac{C_D v_{gr}}{\delta I}.
$$

(22)

Equation (21) may be solved iteratively using a Newton-Raphson algorithm. If the boundary layer depth $\delta$ is prescribed as a function of radius as in section 4.5, the vertical velocity at the top of the layer can be obtained from the continuity equation (15).