* a application of time series analysis

6308

MET? (time series): O'Brien (1 hour, 1995)

- [This question was not specifically covered in any class. It is designed to determine your breath of knowledge of meteorological time scales. You are to be commended for being innovative even if you are naïve.]

Recently an amazing discovery has been made. In Kentucky, near the home of Daniel Boone, a rather remarkable family with the name of Smith settled in 1785. The original grand old person was fascinated by the weather and began daily measurements of precipitation. In spite of bad times, the family over more than 10 generations maintained the measurement gathering. Due to family problems, there are some gaps in the records.

You become aware of the data. Wow! What would you do with this record?

a) Discuss what interesting scientific problems you might address with the data.
b) Discuss what societal problems you might address.
c) What technical problems (and solutions) do the gaps impose?

Question from J. O'Brien.

Estimated time: 45 min.

An oceanographer presenting a paper at a conference on actual measurements of oceanic turbulence might use the following terms. Explain in works or by formula each term below. Be brief.

✓ a. isotropic turbulence
✓ b. homogeneous turbulence
✓ c. rotary spectra
✓ d. inertial subrange
✓ e. Obukhov length scale L
✓ f. co-spectrum
✓ g. quadrature spectrum
✓ h. coherence
✓ i. eddy Prandtl number
* j. bi-spectrum
* ✓ k. log-linear profile
* ✓ l. bulk-aerodynamic formula

O

O'Brien

1 HR

1. Atmospheric Turbulence & Time Series

In 3-D turbulence there exists a relationship between the kinetic energy spectrum and wavenumber called the "minus Five Thirds Law." For large scale atmospheric flow the kinetic energy spectrum of horizontal flow is steeper. If we calculate the kinetic energy spectrum around a latitude circle in mid-troposphere we find something like a "minus 3 law."

A. Assume that the region between wavenumber 10 and 100 depends only on wavenumber, k, and the dissipation of mean-square vorticity, N. Use dimension analysis to derive the minus three law.

B. Suppose you had gridded global weather data of horizontal velocity, list the steps you would use to calculate estimates of the kinetic energy spectrum as a function of east-west wavenumber. Assume you have many realizations so you can use ensemble averaging to increase the degrees of freedom.

C. Explain with a diagram how you would plot the spectrum to demonstrate the -3 law for the calculations in B.

D. Outline how you might use the K-S test to test the hypothesis that the calculations in B were similar or not similar to a -3 spectrum.

O'Brien

* basic dynamics (vector)

MET? : O'Brien (30 minutes)

- Given the "role" for any vector $\mathbf{A}$

$$\left.\frac{d\mathbf{A}}{dt}\right|_{abs} = \left.\frac{d\mathbf{A}}{dt}\right|_{rel} + \Omega \times \mathbf{A}$$

in a rotating system with constant angular spin, Ω. Drive an expression for the acceleration vector for an inertial frame of reference in terms of the acceleration vector relative to the earth plus others terms.
Give a physical explanation of each term in the equation you have derived.

see Holton or next question

* time rate of change of a vector

MET? : O'Brien (30 minutes)

- The relationship between the time rate of change of a vector in an inertial system and the time rate of change of the same vector in a moving system relative to the earth is

$$\frac{d_a\mathbf{A}}{dt} = \frac{d\mathbf{A}}{dt} + \Omega \times \mathbf{A} \qquad (1)$$

Derive the relationship

$$\frac{d_a\mathbf{U}_a}{dt} = \frac{d\mathbf{U}}{dt} + 2\Omega \times \mathbf{U} + \Omega \times (\Omega \times \mathbf{r})$$

Explain all steps.

Sol)

$$\frac{d_a A}{dt} = \frac{dA}{dt} + \Omega \times A \quad (1)$$

time rate change in A in an inertial reference frame = time rate change in A in a noninertial reference frame + rotation of the vector A due to the rotation of the noninertial reference frame.

Let r be a position vector in an inertial reference frame. Then

$$\frac{d_a r}{dt} = \frac{dr}{dt} + \Omega \times r \longrightarrow U_a = U + \Omega \times r$$

But $\frac{d_a r}{dt} = U_a \equiv$ velocity of the tip of r in the inertial reference frame.

$\frac{dr}{dt} = U \equiv$ " " " " " non-inertial " " .

We apply (1) again to U_a

$$\frac{d_a U_a}{dt} = \frac{dU_a}{dt} + \Omega \times U_a$$

$$= \frac{d}{dt}(U + \Omega \times r) + \Omega \times (U + \Omega \times r)$$

$$= \frac{dU}{dt} + \frac{d}{dt}(\Omega \times r) + \Omega \times U + \Omega \times (\Omega \times r)$$

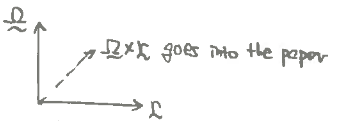
$$\frac{d_a U_a}{dt} = \frac{dU}{dt} + \frac{d\Omega}{dt} \times r + \Omega \times \frac{dr}{dt} + \Omega \times U + \Omega \times (\Omega \times r)$$

Notes

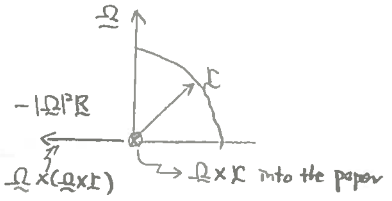
① we assume Ω is a time invariant vector as $\frac{d\Omega}{dt} = 0$

② we defined $\frac{dr}{dt} = U$

③ $\Omega \times r$ is a vector orthogonal to the plane formed by Ω and r



If we then form $\Omega \times (\Omega \times r)$ it is orthogonal to the plane of Ω and r to the right, the $\Omega \times (\Omega \times r)$ is orthogonal to the plane of Ω and $\Omega \times r$ to the right. This makes $\Omega \times \Omega \times r$ points normal to the left of Ω



$$\therefore \frac{d_a U_a}{dt} = \frac{dU}{dt} + 2\Omega \times U + \Omega \times (\Omega \times r)$$

if on the earth,
$\Omega \times \Omega \times r = -|\Omega|^2 R$
where R is the radial vector pointing out and normal to Ω.

★ basic dynamics (geostrophic, thermal wind, barotropic)

MET? : O'Brien (30 minutes) *

- Question

(a) Write down the geostrophic wind equation in component form for pressure coordinates. Explain all symbols.

(b) Derive the thermal wind equation in pressure coordinates.

(c) Define what is meant by a barotropic atmosphere.

(d) Prove the theorem: The geostrophic wind is independent of pressure (height) in a barotropic atmosphere.

(a)

$$\mathbf{V}_g = \frac{1}{f}\mathbf{k} \times \boxed{\nabla_p \Phi}$$

$\nabla_p \Phi$ → gradient of the geopotential $d\Phi = g dz$ on an isobaric sfc

f → Coriolis parameter $f = 2\Omega \sin\phi$, ϕ → latitude

In component form

$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y} \quad ; \quad v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x}$$

(b) Thermal wind ⇒ vertical shear of geostrophic wind.

$$\frac{\partial \mathbf{V}_g}{\partial \ln p} = \frac{1}{f}\mathbf{k} \times \nabla_p \frac{\partial \Phi}{\partial \ln p}$$

Assume a hydrostatic atmosphere, then $\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{RT}{p}$ or $\frac{\partial \Phi}{\partial \ln p} = -RT$

Thus,

$$\frac{\partial \mathbf{V}_g}{\partial \ln p} = -\frac{R}{f}\mathbf{k} \times \nabla_p T \quad : \text{Thermal wind eq.}$$

→ component eq

$$\left(\frac{\partial u_g}{\partial \ln p} = +\frac{R}{f}\frac{\partial T}{\partial y} \quad ; \quad \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f}\frac{\partial T}{\partial x} \right)$$

$$\rightarrow \int_{p_2}^{p_1} \frac{\partial \mathbf{V}_g}{\partial \ln p} d\ln p = -\frac{R}{f}\int_{p_2}^{p_1} \mathbf{k} \times \nabla T \, d\ln p$$

p_1 ← lower level

p_2 ← higher level

$$\mathbf{V}_g(p_1) - \mathbf{V}_g(p_2) = -\frac{R}{f}\int_{p_2}^{p_1} \mathbf{k} \times \nabla T \, d\ln p$$

$$\mathbf{V}_T = \mathbf{V}_g(p_2) - \mathbf{V}_g(p_1) = \frac{R}{f}\int_{p_2}^{p_1} \mathbf{k} \times \nabla T \, d\ln p$$

→ thermal wind is vector difference between upper & lower level geostrophic wind ∝ layer mean temperature gradient.

(c), (d)

A barotropic atmosphere is one in which density is a function of pressure alone. $\rho = \rho(p)$

Thus if we consider the eq. of state

$$p = \rho R T$$

in a barotropic atmosphere we see that

$$\nabla_p p = \nabla_p(\rho R T)$$

$$0 = \rho R \nabla_p T$$

$$0 = \nabla_p T \longrightarrow \text{the horizontal temperature gradient on an isobaric sfc in a barotropic atmosphere is zero}$$

Therefore in a barotropic atmosphere

$$\frac{\partial \mathbf{V}_g}{\partial \ln p} = -\frac{R}{f}\mathbf{k} \times \overset{0}{\cancel{\nabla_p T}} = 0$$

That is, $\mathbf{V}_g$ is independent of height.

* pressure change due to adv.

MET? : O'Brien (30 minutes)

- A balloon flying eastward over Tallahassee and being advected by the wind which is 20 m/s does not measure any pressure change. An observer on the ground measures a change of 0.1 kPa/3 hours.

a) What is the change of pressure due to advection?

b) Is there enough information in this problem to tell in which direction is the nearest high pressure system? Explain your answer.

a) We are given ① $\frac{\partial p}{\partial t} = 0.1\ \text{kPa}/3\text{hrs}$

② $\frac{dp}{dt} = 0$ where $\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$

$v = 0,\ w = 0,\ u = 20\text{m/s}$

So $0 = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}$

$-u\frac{\partial p}{\partial x} = \frac{\partial p}{\partial t} = \frac{0.1\text{kPa}}{8\text{hrs}}$

$\frac{\partial p}{\partial x} = -\frac{0.1\text{kPa}}{3\text{hrs}} \cdot \frac{1}{20\text{m/s}}$

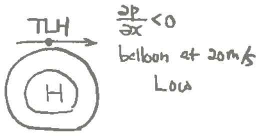
$\hookrightarrow$ pressure decreases heading east.

$\Rightarrow$ pressure Change due to adv. is $\frac{0.1\text{kPa}}{3\text{hrs}}$

b)

$\frac{\partial p}{\partial t} = 0.1\ \text{kPa}/3\text{hrs}$

Since $\frac{dp}{dt} = 0$ the balloon is flying along an isobar. An westerly wind occurs on the northern edge of a H in the northern Hemisphere The pressure decreases moving east of the observer. These two facts place high pressure South of TLH



For O'Brien; 45-60 minutes.

Radiosondes are used for vertical sampling of atmospheric structure on a routine basis. Unfortunately, the sampling density is small. This can be overcome to some extent by demanding that the data satisfy one or more equations of constraint.

Suppose you are given preliminary analyses of 500-mb geopotential heights and winds on a regularly-spaced net of points – say in the continental United States. Moreover, it appears from the preliminary (independent) analyses that the winds and heights nearly satisfy the geostrophic condition. Following the Sasaki approach, formula (this is to include clear statements of the underlying philosophy) a variational adjustment procedure that will guarantee exact satisfaction of geostrophy. Find the Euler-Lagrange equations and describe the manner of solution. Use rectangular cartesian coordin

O'Brien

30 min. 2) Explain the significance of the Väisalä-Brunt Frequency in the dynamics of a stratified fluid (stably stratified). First define the quantity in a precise way.

30 min. 3) Explain how the von Neumann linear stability criteria is used to determine if predictive finite difference equations are numerically stable. Carefully explain the concepts of stability and convergence in this context and give a simple example to describe the above test.

'Brien 2. Numerical Weather Prediction (O'Brien)

min.

a. Show that upstream differencing is a dissipative finite-difference scheme. Derive an approximate value for the implicit diffusion coefficient. Show that the leap-frog scheme conserves energy if the CFL condition is satisfied. Hint: Use a 1-D linear advection equation.

b. Semi-implicit schemes are becoming widely-used in meteorology and oceanography. Describe, using a simple set of of equations, the basic idea for these schemes.

* Stability + finite difference approx. in NWP

5541

MET? (NWP): O’Brien (?, 1997)

- Please work enough problems to equal 100 pts. Only.

1) Carefully define the terms Neutral, Stable, Unstable with regard to the amplification factor for linear stability.
2) Consider the 2-D diffusion equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

   a) Write down the finite difference approximation for forward in time and centered in space.
   b) Derive the condition for which this scheme is stable.

* Stability

MET$\overset{5541}{?}$: O'Brien (?)

- Consider the equation

$$\frac{\partial w}{\partial t} + \hat{A}\frac{\partial w}{\partial x} = \hat{K}\frac{\partial^2 w}{\partial x^2} - i\,\hat{f}\,w$$

where $w = u + iv$

and $\hat{A}$, $\hat{K}$, $\hat{f}$ and are arbitrary constants.

For the following special cases, indicate (writhe down)

(a) a stable scheme and its limiting condition, if any

(b) an unconditionally unstable scheme.

The special conditions are

(A) $\hat{A} = \hat{K} = 0$

(B) $\hat{A} = \hat{f} = 0$

(C) $\hat{K} = \hat{f} = 0$

* Stability

MET$\overset{5541}{?}$ (NWP): O'Brien (30 minutes) *

- Consider the linearized equations of motion

$$\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}=fv \qquad (1)$$

$$\frac{\partial v}{\partial t}+A\frac{\partial v}{\partial x}=-fu \qquad (2)$$

for which the following numerical scheme is suggested

$$u_j^{n+1}-u_j^n=-\lambda\left(\overline{u_x}^x\right)+f'v_j^n \qquad (3)$$

$$v_j^{n+1}-v_j^n=-\lambda\left(\overline{v_x}^x\right)-f'u_j^n \qquad (4)$$

where $\lambda = A\Delta t/\Delta x$ and $f' = f\Delta t$ and

$$\overline{w_\alpha}^\alpha \equiv \frac{1}{2}\left[w_{j+1}^n - w_{j-1}^n\right]$$

(a) Use the von Neumann stability test to determine under what conditions this scheme is stable.

(b) Suggest a finite difference scheme which is stable for all Δt and Δx.

(a) We first rewrite (3) + (4) with the shorthand notation $\overline{U_x}^x$ and $\overline{V_x}^x$ removed

$$U_j^{n+1}=U_j^n-\frac{A\Delta t}{2\Delta x}(U_{j+1}^n-U_{j-1}^n)+\Delta t f V_j^n \qquad (5)$$

$$V_j^{n+1}=V_j^n-\frac{A\Delta t}{2\Delta x}(V_{j+1}^n-V_{j-1}^n)-\Delta t f U_j^n \qquad (6)$$

Let $U_j^n = U^n e^{ikj\Delta x}$

$V_j^n = V^n e^{ikj\Delta x}$

Substitute these into (5) + (6)

$$U^{n+1}\cancel{e^{ikj\Delta x}}=U^n e^{ikj\Delta x}-\frac{A\Delta t}{2\Delta x}\left(U^n e^{ik(j+1)\Delta x}-U^n e^{ik(j-1)\Delta x}\right)+\Delta t f V^n e^{ikj\Delta x}$$

$$U^{n+1}=U^n-\frac{A\Delta t}{2\Delta x}U^n(2i\sin k\Delta x)+\Delta t f V^n$$

$$\boxed{U^{n+1}=U^n-i\left(\frac{A\Delta t}{\Delta x}\right)U^n\sin k\Delta x+\Delta t f V^n} \qquad (7)$$

Similarly for eq. (6)

$$\boxed{V^{n+1}=V^n-i\left(\frac{A\Delta t}{\Delta x}\right)\sin k\Delta x V^n-\Delta t f U^n} \qquad (8)$$

To simplify what follows that

$$\alpha=i\left(\frac{A\Delta t}{\Delta x}\right)\sin k\Delta x \quad ; \quad \beta=f\Delta t$$

With these parameters (7) + (8) become

$$\begin{cases} U^{n+1}=U^n-\alpha U^n+\beta V^n \\ V^{n+1}=V^n-\alpha V^n-\beta U^n \end{cases}$$

or

$$\begin{cases} U^{n+1}=(1-\alpha)U^n+\beta V^n & (9) \\ V^{n+1}=(1-\alpha)V^n-\beta U^n & (10) \end{cases}$$

We need to combine (9) + (10) somehow to get an eq. solely in U or V

Toward this end we divide (9) by β

$$\frac{1}{\beta}U^{n+1}=\frac{1-\alpha}{\beta}U^n+V^n \longrightarrow \text{multiply by } (1-\alpha):\ \underbrace{\frac{(1-\alpha)}{\beta}U^{n+1}=\frac{(1-\alpha)^2}{\beta}U^n+(1-\alpha)V^n}$$

Now increment the time index by one

$$\boxed{\frac{1}{\beta}U^{n+2}=\frac{(1-\alpha)}{\beta}U^{n+1}+V^{n+1}}$$

↳ Subtract these two equations

replace this using (10)

$$\frac{1}{\beta}U^{n+2}-\frac{1-\alpha}{\beta}U^{n+1}=\frac{1-\alpha}{\beta}U^{n+1}-\frac{(1-\alpha)^2}{\beta}U^n+\boxed{V^{n+1}-(1-\alpha)V^n}$$

$$\frac{1}{\beta}U^{n+2}-\frac{1-\alpha}{\beta}U^{n+1}=\frac{1-\alpha}{\beta}U^{n+1}-\frac{(1-\alpha)^2}{\beta}U^n-\beta U^n$$

multiply by β and group like terms

$$U^{n+2}-2(1-\alpha)U^{n+1}+[(1-\alpha)^2+\beta^2]U^n=0$$

Define the amplitude factor λ such that $U^{n+1}=\lambda U^n$

Substitute this into the above eq.

$$\lambda^2-2(1-\alpha)\lambda+[(1-\alpha)^2+\beta^2]=0$$

We must solve for the eigenvalues (roots) of the above quadratic eq.

$$\lambda=(1-\alpha)\pm\sqrt{(1-\alpha)^2-[(1-\alpha)^2+\beta^2]}$$

$$=(1-\alpha)\pm i\beta$$

We have the two roots

$$\lambda_1=1-\alpha+i\beta \quad ; \quad \lambda_2=1-\alpha-i\beta$$

Recall

$$\alpha=i\left(\frac{A\Delta t}{\Delta x}\right)\sin k\Delta x \ , \ \beta=f\Delta t$$

Let's replace α by ir where $r=\frac{A\Delta t}{\Delta x}\sin k\Delta x$

Then $\lambda_1=1+i(\beta-r)$; $\lambda_2=1-i(\beta+r)$

We want the magnitudes

$$|\lambda_1|=(\lambda_1\lambda_1^*)^{1/4}=(1+(\beta-r)^2)^{1/2}=(1+\beta^2-2\beta r+r^2)^{1/2}$$

$$|\lambda_2|=(1+(\beta+r)^2)^{1/2}$$

???

We see that the roots λ_1 and λ_2 have the same magnitude, they are only in phase. As magnitude is a positive definite quantity we must have

$$(\beta-r)^2\geq 0$$

$$\beta-r\geq 0$$

$$\beta\geq r$$

$$f\Delta t\geq\frac{A\Delta t}{\Delta x}\sin kx \Rightarrow 1\geq\frac{A}{f\Delta x}\sin k\Delta x$$

But $|\sin k\Delta x|\leq 1$ so

$$\underline{\frac{A}{f\Delta x}\leq 1}$$

is the condition for stability of the scheme.

- Apply the <u>matrix method</u> to this problem. We first rewrite the system of eqs as

$$\begin{cases} (1-\alpha)U^n+\beta V^n=U^{n+1} \\ -\beta U^n \;+(1-\alpha)V^n=V^{n+1} \end{cases}$$

or

$$\underbrace{\begin{pmatrix}(1-\alpha) & \beta \\ -\beta & (1-\alpha)\end{pmatrix}}\underbrace{\begin{pmatrix}U^n \\ V^n\end{pmatrix}}_{F^n}=\underbrace{\begin{pmatrix}U^{n+1} \\ V^{n+1}\end{pmatrix}}_{F^{n+1}} \quad\Rightarrow$$

↳ we call this the amplification matrix

It is apparent that the question of computational stability is related to the properties of the matrix A

$AF^{n-1}=F^n$ so

$\rightarrow AF^n=A^2F^{n-1}=F^{n+1}$

$AF^n=F^{n+1}$

so

$AF^{n+1}=F^{n+2}$

$A(AF^n)$

$\boxed{A^2}F^n=F^{n+2}$

↳ whether the solution is stable or unstable depends (somehow) on A

The so-called eigenvalues (characteristic values) λ_k of A are the roots of the characteristic eq

$$|A-\lambda I|=0$$

↖ identity matrix

Associated with each eigenvalue λ_k is an eigenvector V_k which satisfies the eq.

$$AV_k=\lambda_k V_k \quad \text{for } k=1,2,\cdots$$

When the eigenvectors form a complete, linear independent set, an arbitrary initial condition F_0 can be expressed as a linear combination of the eigenvectors

$$F_0=\sum_k C_k V_k \qquad \text{where } C_k \text{ are constants.}$$

<u>Sufficient condition</u>

A sufficient condition for the matrix A to have a complete set of eigenvectors is that the eigenvalues be nonzero and differ from one another. However, under certain circumstances a repeated eigenvalue may also lead to a complete set of linearly independent vectors.

Real symmetric matrices, Hermitian matrices, normal matrices lead to linearly independent sets. Assuming the sufficient condition is fulfilled,

$$F_0=\sum_k C_k V_k$$

and

$$F_n=A^nF_0$$

imply

$$F_n=A^n\sum_k C_kV_k=\sum_k C_kA^nV_k$$

$$F_n=\sum_k C_kA^{n-1}(AV_k)$$

$$F_n=\sum_k C_kA^{n-1}(\lambda_kV_k)$$

Repeat this process to A^0

$$\boxed{F_n=\sum_k C_k\lambda_k^nV_k}$$

It is evident that the solution F_n remains bounded provided that the eigenvalues have magnitudes less than or equal to one. That is.

$$|\lambda_k|\le 1 \quad \text{for all } k.$$

In general, a particular nonzero eigenmode will amplify, remain neutral, or dampen according to whether its associated eigenvalue has a magnitude greater, equal, or less than one

$\lambda_k>1$, amplification of k^{th} eigenmode

$|\lambda_k|=1$, neutral

$\lambda_k<1$, damping

Strictly speaking, the condition $|\lambda_k|\le 1$ for all k is more stringent than is required for stability according to some definitions of stability. The eigenvalues may be permitted to exceed one by a term having magnitude no larger than order Δt. That is,

$$|\lambda_k|\le 1+O(\Delta t) \quad \text{for all } k.$$

is "stable"

This condition permits exponential growth of the solution <u>but</u> the solution remains bounded for a fixed time interval T. Such amplification may be fit + proper for the physical-mathematical system with, say, baroclinic instabili

So we have the matrix eq.

$$\begin{pmatrix}(1-\alpha) & \beta \\ -\beta & (1-\alpha)\end{pmatrix}\begin{pmatrix}U^n \\ V^n\end{pmatrix}=\begin{pmatrix}U^{n+1} \\ V^{n+1}\end{pmatrix}$$

or

$$AF_n=F_{n+1}$$

We need the eigenvalues of A. They are the roots to the characteristic eq

$$|A-\lambda I|=0$$

$$\begin{vmatrix}(1-\alpha)-\lambda & \beta \\ -\beta & (1-\alpha)-\lambda\end{vmatrix}=0$$

$$(1-\alpha)^2-2(1-\alpha)\lambda+\lambda^2+\beta^2=0$$

or

$$\lambda^2-2(1-\alpha)\lambda+(1-\alpha)^2+\beta^2=0$$

Solve for λ

$$\lambda=(1-\alpha)\pm\sqrt{(1-\alpha)^2-(1-\alpha)^2-\beta^2}$$

$$=(1-\alpha)\pm i\beta$$

Thus, the two eigenvalues are

$$\lambda_1=1-\alpha+i\beta \;;\; \lambda_2=1-\alpha-i\beta$$

We are going to make another change in notation since α is complex.

Define γ such that

$$i\frac{A\Delta t}{\Delta x}\sin k\Delta x=\alpha=i\gamma \qquad \text{so} \quad \gamma=\frac{A\Delta t}{\Delta x}\sin k\Delta x$$

Then

$$\lambda_1=1+i(\beta-\gamma) \quad \text{and} \quad \lambda_2=1-i(\beta+\gamma)$$

Of interest here is the magnitude of the eigenvalues. For stability $|\lambda_1|\le 1$, |

Thus,

$$|\lambda_1|=\left[1+(\beta-\gamma)^2\right]^{1/2}\le 1 \quad \text{and} \quad |\lambda_2|=\left[1+(\beta-\gamma)^2\right]^{1/2}\le 1 \quad \longleftarrow ???$$

Note that the two eigenvalues have the same magnitude. They differ only in ph

Since magnitude is real we want

$$\beta\ge\gamma$$

$$f\Delta t\ge\frac{A\Delta t}{\Delta x}\sin k\Delta x$$

$$1\ge\frac{A}{f\Delta x}\sin k\Delta x$$

As $|\sin k\Delta x|\le 1$ the above condition will be met if

$$\frac{A}{f\Delta x}\le 1$$

We conclude that the scheme will be stable provided

$$\frac{A}{f\Delta x}\le 1$$

Note that f has units of inverse time. Therefore, we could define a Coriolis time scale as $\Delta t_f=f^{-1}$. The above condition becomes

$$\frac{A\Delta t_f}{\Delta x}\le 1$$

This is the familiar CFL condition with $A\equiv$ phase speed of wave.

(b)

We can make the scheme stable for all Δt and Δx as follows. We have the stabil criterion $\frac{A}{f\Delta x}\le 1$. Clearly if $A=0$, the scheme will be unconditionally stable. With $A=0$ we remove the nonlinear advection terms. We are left with purely motions which are unconditionally stable. A common way to make the finite dif form of a term unconditionally stable is to use <u>an implicit scheme</u>. Thus we re

$$U^n_{j+1}-U^n_{j-1}$$

with

$$U^n_{j+1}-U^n_{j-1}=\frac{U^{n+1}_{j+1}+U^n_{j+1}}{2}-\frac{U^{n+1}_{j-1}+U^n_{j-1}}{2}$$

Similarly with

$$V_{j+1}^n - V_{j-1}^n = \frac{V_{j+1}^{n+1} + V_{j+1}^n}{2} - \frac{V_{j-1}^{n+1} + V_{j-1}^n}{2}$$

With these changes (5) + (6) becomes

$$\begin{cases} u_j^{n+1} = u_j^n - \frac{A\Delta t}{2\Delta x}\left(\frac{u_{j+1}^{n+1}+u_{j+1}^n}{2} - \frac{u_{j-1}^{n+1}+u_{j-1}^n}{2}\right) + f\Delta t\, V_j^n \\ V_j^{n+1} = V_j^n - \frac{A\Delta t}{2\Delta x}\left(\frac{V_{j+1}^{n+1}+V_{j+1}^n}{2} - \frac{V_{j-1}^{n+1}+V_{j-1}^n}{2}\right) - f\Delta t\, u_j^n \end{cases}$$

Again let $u_j^n = U^n e^{ikj\Delta x}$

$V_j^n = V^n e^{ikj\Delta x}$

Then substitute into preceding system of eqs.

$$\begin{cases} U^{n+1} = U^n - \frac{A\Delta t}{2\Delta x}\left(\frac{1}{\cancel{2}}\right)\left(U^{n+1}\cancel{2}i\sin k\Delta x + U^n\cancel{2}i\sin k\Delta x\right) + f\Delta t\, V^n \\ V^{n+1} = V^n - \frac{A\Delta t}{2\Delta x}\left(\frac{1}{\cancel{2}}\right)\left(V^{n+1}\cancel{2}i\sin k\Delta x + V^n\cancel{2}i\sin k\Delta x\right) - f\Delta t\, U^n \end{cases}$$

$$\begin{cases} U^{n+1} = U^n\left(1 - \frac{A\Delta t}{2\Delta x} i\sin k\Delta x\right) - \frac{A\Delta t}{2\Delta x} i\sin k\Delta x\, U^{n+1} + f\Delta t\, V^n \\ V^{n+1} = V^n\left(1 - \frac{A\Delta t}{2\Delta x} i\sin k\Delta x\right) - \frac{A\Delta t}{2\Delta x} i\sin k\Delta x\, V^{n+1} - f\Delta t\, U^n \end{cases}$$

$$\begin{cases} \left(1 + i\frac{A\Delta t}{2\Delta x}\sin k\Delta x\right)U^{n+1} = \left(1 - i\frac{A\Delta t}{2\Delta x}\sin k\Delta x\right)U^n + f\Delta t\, V^n \\ \left(1 + i\frac{A\Delta t}{2\Delta x}\sin k\Delta x\right)V^{n+1} = \left(1 - i\frac{A\Delta t}{2\Delta x}\sin k\Delta x\right)V^n - f\Delta t\, U^n \end{cases}$$

Let $\alpha = \frac{A\Delta t}{2\Delta x}\sin k\Delta x$, $\beta = f\Delta t$

$$\begin{cases} (1+i\alpha)U^{n+1} = (1-i\alpha)U^n + \beta V^n \\ (1+i\alpha)V^{n+1} = (1-i\alpha)V^n - \beta U^n \end{cases}$$

or

$$U^{n+1} = (1-i\alpha)(1+i\alpha)^{-1}U^n + \beta(1+i\alpha)^{-1}V^n$$

$$V^{n+1} = (1-i\alpha)(1+i\alpha)^{-1}V^n - \beta(1+i\alpha)^{-1}U^n$$

Rewrite this as a matrix eq.

$$\begin{pmatrix} U^{n+1} \\ V^{n+1} \end{pmatrix} = \begin{pmatrix} (1-i\alpha)(1+i\alpha)^{-1} & \beta(1+i\alpha)^{-1} \\ -\beta(1+i\alpha)^{-1} & (1-i\alpha)(1+i\alpha)^{-1} \end{pmatrix}\begin{pmatrix} U^n \\ V^n \end{pmatrix}$$

∵ Note that the quotient

$$\frac{1-i\alpha}{1+i\alpha} = \frac{(1)(1)+(-\alpha)(+\alpha)}{1^2+\alpha^2} + i\,\frac{((-\alpha)-(1)(\alpha)}{1^2+\alpha^2}$$

$$= \frac{1-\alpha^2}{1+\alpha^2} + i\,\frac{-2\alpha}{1+\alpha^2} = (1+\alpha^2)^{-1}\left[(1-\alpha^2) - 2\alpha i\right]$$

Eigenvalues of the coefficient matrix are roots of the eq

$$\left((1-i\alpha)(1+i\alpha)^{-1} - \lambda\right)^2 + \left[\beta(1+i\alpha)^{-1}\right]^2 = 0$$

$$\lambda^2 - 2(1-i\alpha)(1+i\alpha)^{-1}\lambda + (1-i\alpha)^2(1+i\alpha)^{-2} + \beta^2(1+i\alpha)^{-2} = 0$$

$$\lambda^2 - 2(1-i\alpha)(1+i\alpha)^{-1}\lambda + \left((1-i\alpha)^2+\beta^2\right)(1+i\alpha)^{-2} = 0$$

$$\lambda = (1-i\alpha)(1+i\alpha)^{-1} \pm \sqrt{(1-i\alpha)^2(1+i\alpha)^{-2} - \left((1-i\alpha)^2+\beta^2\right)(1+i\alpha)^{-2}}$$

$$\lambda = (1-i\alpha)(1+i\alpha)^{-1} \pm i\beta(1+i\alpha)^{-1}$$

$$\lambda = (1+i\alpha)^{-1}\left[(1-i\alpha) \pm i\beta\right]$$

$$\lambda_1 = (1+i\alpha)^{-1}\left[1 - i(\alpha-\beta)\right] \quad \text{or} \quad \lambda_2 = (1+i\alpha)^{-1}\left[1 - i(\alpha+\beta)\right]$$

Of interest is the maximum eigenvalue

$$|\lambda_1| = \frac{1+(\alpha-\beta)^2}{1+\alpha^2} = \frac{1+\alpha^2-2\alpha\beta+\beta^2}{1+\alpha^2} = 1 + \frac{\beta(\beta-2\alpha)}{1+\alpha^2}$$

$$|\lambda_2| = \frac{1+(\alpha+\beta)^2}{1+\alpha^2} = \underbrace{\boxed{1 + \frac{\beta(\beta+2\alpha)}{1+\alpha^2}}}_{\text{max eigenvalue}}$$

Recall $\beta = f\Delta t$, $\alpha = \frac{A\Delta t}{2\Delta x}\sin k\Delta x$

For stability in the von Neumann sense we need $|\lambda_{max}| \le 1 + O(\Delta t)$

we have

$$|\lambda_2| = 1 + \frac{(f\Delta t)(f\Delta t) + 2(f\Delta t)\left(\frac{A\Delta t}{2\Delta x}\sin k\Delta x\right)}{1 + \left(\frac{A\Delta t}{2\Delta x}\sin k\Delta x\right)^2}$$

Stability requires

$$|\lambda_2| = 1 + \frac{f^2(\Delta t)^2 + \frac{fA(\Delta t)^2}{\Delta x}\sin k\Delta x}{1 + \frac{A^2(\Delta t)^2}{(\Delta x)^2}\sin^2 k\Delta x} \le 1 + O(\Delta t)$$

⋮

mess !

* boundary value problem

MET? : O'Brien (35 minutes)

- We wish to solve the boundary value problem

$$-u_{xx} - u_{yy} + c(x,y)u(x,y) = f(x,y) \qquad (1)$$

on a rectangle D:

$$0 \le x \le 1, \qquad 0 \le y \le 1$$

subject to the boundary conditions

$$\left.\begin{aligned} \frac{\partial u(0,y)}{\partial n} &= h_1(y) \\ u(1,y) &= h_2(y) \end{aligned}\right\} 0 \le y \le 1$$

$$\left.\begin{aligned} u(x,0) &= h_3(x) \\ u(x,1) &= h_4(x) \end{aligned}\right\} 0 \le x \le 1$$

Here, h_i, c and f are known functions, and $\partial / \partial n$ denotes the outward normal derivative.

a) If $c < 0$, what difficulty might arise when attempting to solve this problem?

b) Assume $c \ge 0$. Impose a uniform rectangular mesh on the region. Let N be a positive integer, $h = 1/N$ be the mesh size, $x_j = jh$, $j = 0,1,\ldots,N$ and $y_k = kh$, $k = 0,1,\ldots,N$ and let u_{jk} be the numerical approximation to $u(x_j, y_k)$. Describe the 5-point approximation to (1) in the interior of the region.

c) Assume that $u(x, y)$ is four times continuously differentiable in both variables. Use Taylor series to derive the local truncation error for the difference operator in (b). What is the order of the local truncation error as $h \to 0$?

d) Give two different approximations to the boundary condition at the left boundary. Also state the order (as $h \to 0$) of the local truncation error of each.

★ simple barotropic weather model

MET$\overset{5541}{?}$ (Numerical analysis): O'Brien (1 hour)

- The equations for a simple barotropic weather model are

$$\frac{\partial\zeta}{\partial t} + \mathbf{V}_g\cdot\nabla(\zeta_g + f) = 0$$

$$\mathbf{V}_g = \mathbf{k}\times\nabla\phi / f_0 \quad = -\frac{1}{f_0}\frac{\partial\phi}{\partial y}\mathbf{i} + \frac{1}{f_0}\frac{\partial\phi}{\partial x}\mathbf{j}$$

$$\nabla^2\phi / f_0 = \zeta \quad = \frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \longrightarrow \nabla(\zeta_g + f) = \frac{1}{f_0}\frac{\partial}{\partial x}\nabla^2\phi\,\mathbf{i} + \frac{1}{f_0}\frac{\partial}{\partial y}\nabla^2\phi\,\mathbf{j} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$f = f_0 + \beta y$$

Suppose you solve this in a β–channel with given initial condition $\phi(x,y,0) = A(x,y)$. Write the problem in a stable finite difference model. Specify all boundary conditions and parameters. Suggest a numerical method to calculate a 24 hour forecast with Δt = 1 hour. In other words, convince me that you could give this problem in great detail to be done by an undergraduate student on a computer!

Sol)

We will first expand the term

$$V_g\cdot\nabla(\zeta_g + f) = u_g\frac{\partial}{\partial x}(\zeta_g + \cancel{f}) + v_g\frac{\partial}{\partial y}(\zeta_g + f)$$

f only varies in y

$$= u_g\frac{\partial}{\partial x}\left(\frac{1}{f_0}\nabla^2\phi\right) + v_g\frac{\partial}{\partial y}\left(\frac{1}{f_0}\nabla^2\phi\right) + v_g\left(\frac{\partial f}{\partial y}\right)$$

on a β plane $\frac{\partial f}{\partial y} = \beta$

$$= \left(-\frac{1}{f_0}\frac{\partial\phi}{\partial y}\right)\frac{\partial}{\partial x}\left(\frac{1}{f_0}\nabla^2\phi\right) + \left(\frac{1}{f_0}\frac{\partial\phi}{\partial x}\right)\frac{\partial}{\partial y}\left(\frac{1}{f_0}\nabla^2\phi\right) + \beta v_g$$

$$= \frac{1}{f_0^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\phi) - \frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\phi)\right) + \beta v_g$$

$$= \frac{1}{f_0^2}J(\phi, \nabla^2\phi) + \beta v_g$$

where $J(\alpha,\beta) = \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial x}$ is the Jacobian operator.

We use this to rewrite the barotropic vorticity eq as

$$\frac{1}{f_0}\frac{\partial}{\partial t}(\nabla^2\phi) + \frac{1}{f_0^2}J(\phi, \nabla^2\phi) + \beta \underset{\uparrow\,\frac{1}{f_0}\frac{\partial\phi}{\partial x}}{v_g} = 0$$

or

$$\frac{\partial}{\partial t}(\nabla^2\phi) + (f_0)^{-1}J(\phi, \nabla^2\phi) + \frac{\partial\phi}{\partial x}\beta = 0$$

or

$$\boxed{\frac{\partial}{\partial t}(\nabla^2\phi) = -(f_0)^{-1}J(\phi, \nabla^2\phi) - \beta\frac{\partial\phi}{\partial x}}$$

This is a prognostic eq. for $\nabla^2\phi$. However in this form it is off little use to use. Define the geopotential tendency $\chi = \frac{\partial\phi}{\partial t}$.

In terms of χ the above becomes

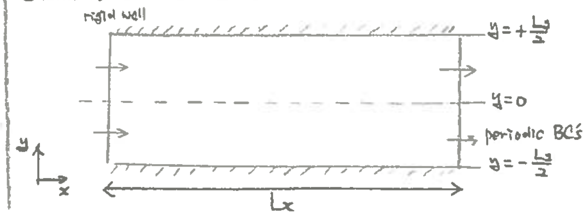
$$\nabla^2\chi + F(\phi, \nabla^2\phi) = 0$$

where

$$F(\phi, \nabla^2\phi) = f_0^{-1}J(\phi, \nabla^2\phi) + \beta\frac{\partial\phi}{\partial x}$$

This is a Poisson eq in χ. If we know χ we can use a finite difference scheme to project to a future ϕ distribution.

We will use relaxation to solve the Poisson eq. But first we must describe the grid. We are integrating the barotropic eq on a β plane channel. Let x point eastward, y northward. The flow is periodic in x. We have walls at $y = \pm\frac{L_y}{2}$ taking $y = 0$ at the center of the channel. Let the channel



be of zonal length Lx. Let there be (M+1) points in the x direction and

points in the y direction. For simplicity use a grid mesh in which $\Delta x = \Delta y = d$

point on the mesh is uniquely located by the indices i and j. The point at (i

corresponds to position $x = id$, $y = jd$ where $i, j = 0, 1, 2, \cdots, \left(\begin{smallmatrix}M\\N\end{smallmatrix}\right.$

We will use centered difference in space. Thus

$$\left.\frac{\partial\phi}{\partial x}\right|_{ij} \simeq \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2d} \quad ; \quad \left.\frac{\partial\phi}{\partial y}\right|_{ij} \simeq \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2d}$$

$$\left.\frac{\partial^2\phi}{\partial x^2}\right|_{ij} \simeq \frac{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}}{d^2} \quad ; \quad \left.\frac{\partial^2\phi}{\partial y^2}\right|_{ij} \simeq \frac{\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}}{d^2}$$

$$\left.\nabla^2\phi\right|_{ij} \simeq \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{ij}}{d^2} \equiv \nabla^2\phi_{ij}$$

↳ we can use this to finite difference $\nabla^2\chi$. Near the N + S boundaries we

use one sided differences for

$$\left.\frac{\partial\phi}{\partial y}\right|_{ij} = \begin{cases} \dfrac{\phi_{i,N} - \phi_{i,N-1}}{d} & \text{at the northern boundary} \\[2ex] \dfrac{\phi_{i,1} - \phi_{i,0}}{d} & \text{at the southern boundary} \end{cases}$$

Due to the periodicity in x.

$$\left.\frac{\partial\phi}{\partial x}\right|_{ij} = \begin{cases} -\dfrac{\phi_{0,j} - \phi_{M-1,j}}{2d} & \text{at } i = M \\[2ex] -\dfrac{\phi_{1,j} - \phi_{M,j}}{2d} & \text{at } i = 0 \end{cases}$$

Similar one sided differences can be applied for the Laplacian.

The lateral BCs along the northern + southern walls must be specified. We

to hold the initial values of ϕ along these boundaries constant throughout the

In this case we don't need the one sided differences along there boundaries.

As discussed by Fjortoft the barotropic model conserves certain domain quantities, these include mean vorticity, mean vorticity squared (enstrophy) mean K.E. We want the Jacobian in the forcing term to satisfy similar (

Arakawa described such a Jacobian (2nd order accurate, 4 point stencil, conserves $\overline{\zeta}$, $\overline{\zeta^2}$, $\overline{KE}$) ⟶ $\frac{1}{3}$ sum of 3 Jacobians

We will use the Arakawa Jacobian

Thus at any grid point on our domain we have the eq.

$$\nabla^2\chi_{ij} + F_{ij} = 0$$

As we stated earlier we solve this using relaxation. Consider the point ij. We expand $\nabla^2\chi_{ij}$

$$\chi_{i+1,j} + \chi_{i-1,j} + \chi_{i,j+1} + \chi_{i,j-1} - 4\chi_{ij} + d^2F_{ij} = 0$$

It is unlikely that the LHS will exactly $=0$. Rather we more likely have

$$\chi_{i+1,j} + \chi_{i-1,j} + \chi_{i,j+1} + \chi_{i,j-1} - 4\chi_{ij} + d^2F_{ij} = \varepsilon_{ij}$$

We define a new value at ij such that

$$\chi_{i+1,j} + \chi_{i-1,j} + \chi_{i,j+1} + \chi_{i,j-1} - 4\chi^*_{ij} + d^2F_{ij} = 0$$

Subtracting these eqs we see

$$-4\chi_{ij} + 4\chi^*_{ij} = \varepsilon_{ij}$$

or $$\chi^*_{ij} = \frac{\varepsilon_{ij}}{4} + \chi_{ij}$$

but $$\varepsilon_{ij} = \chi_{i+1,j} + \chi_{i-1,j} + \chi_{i,j+1} + \chi_{i,j-1} - 4\chi_{ij} + d^2F_{ij}$$

or $$\varepsilon_{ij} = d^2\nabla^2\chi_{ij} + d^2F_{ij}$$

Thus we have an iterative scheme in which a new value for χ at ij is given by

$$\chi^{\nu+1}_{ij} = \chi^{\nu}_{ij} + \frac{1}{4}\varepsilon_{ij}$$

where $$\varepsilon_{ij} = (d^2\nabla^2\chi_{ij} + d^2F_{ij})^{\nu} \xleftarrow{} \text{iteration counter}$$

We iterate until ① we achieve an acceptably small ε_{ij} and/or
② we exceed the maximum number of iterations.

If we sequentially apply this iterative algorithm at each ij marching across the grid we call the method <u>successive relaxation</u>. A slower method of relaxation is to compute all $\chi^{\nu+1}_{ij}$ after all χ^{ν}_{ij} are known. This is <u>simultaneous relaxation</u>. A further increase in speed of convergence can be realized if we introduce a $\binom{\text{under-}}{\text{over-}}$ relaxation parameter $\alpha\binom{\alpha<1}{\alpha>1}$ so that our iteration eq becomes

$$\chi^{\nu+1}_{ij} = \chi^{\nu}_{ij} + \frac{\alpha}{4}\varepsilon_{ij}$$

Once we solve the poisson for $\chi = \frac{\partial\phi}{\partial t}$ we may use this tendency to advance the ϕ field. On the first timestep we use a forward or Euler scheme.

$$\phi^{t+\Delta t}_{ij} = \phi^{t}_{ij} + (\Delta t)\left(\frac{\partial\phi}{\partial t}\right)^t_{ij}$$

For subsequent timesteps we use the leapfrog (or centered in time) scheme

$$\phi^{t+\Delta t}_{ij} = \phi^{t-\Delta t}_{ij} + (2\Delta t)\left(\frac{\partial\phi}{\partial t}\right)^t_{ij}$$

The Euler scheme is unstable but we only apply it for a single timestep. The leapfrog scheme is conditionally stable. The condition is the CFL criterion which in 2D is

$$\frac{\sqrt{2}\,C\Delta t}{d} \leq 1$$

$\xleftarrow{}$ we have let $\Delta x = \Delta y = d$

Δt is our timestep. C is the speed of the fastest moving wave described by our model. Our form of the barotropic model only describes relatively slow moving Rossby waves. These waves move with a phase speed of ~8 m/s for a wavelength of 6000 km. As an upper bound we will take C = 20 m/s. We are told to use a timestep of 1hr. Thus our grid spacing d must be such that

$$\sqrt{2}\,C\Delta t \leq d$$

$$\sqrt{2}\ 20\,\text{m/s}\ (3600\,\text{s}) \leq d$$

$$101.8\,\text{km} \leq d$$

$$d \geq \sim 102\,\text{km}$$

We will take d = 110 km

→ As a rule of thumb you like to have 10 gridpoints to resolve features. Features with scales < 10d should not be view with high confidence. Thus there is a desire to use as small a grid increment as possible. With d = 110 km, we should be able to resolve synoptic scale phenomena of scales ≥ 110 km fairly well (well defined in a barotropic atm.)

Our time tendency term $\frac{\partial\phi}{\partial t}$ is only 1st order in time. The centered in ti finite difference approx. is second order in time. Thus the leapfrog schem two solutions

→ physical solution corresponding to the real world problem

→ computational solution which is a numerical artifact of applying a 2' in time finite difference to a 1st order in time eq. The computational changes sign every timestep and grows in time. We must deal with it in way if we are to run our model for an extended period. Matsuno deriv a scheme to damp the computational mode of leapfrog scheme. Periodically the integration we perform an Euler-backward step.

We make a trial forward step over Δt. Use this to compute a new fo and then take a backward step.

forward ① $F^{(n+1)*}_{ij} = F^{n}_{ij} + \Delta t(\text{forcing at time } n)$

backward ② $F^{n+1}_{ij} = F^{n}_{ij} + \Delta t\left(\text{forcing using } F^{(n+1)*}_{ij}\right)$

The Euler backward scheme has no computational mode, however it is But this is exactly what we want in apply this scheme periodically wit our leapfrog time differenced barotropic model

<u>Summary of numerical integration of barotropic vorticity eq</u>

① Read in initial geopotential field at all gridpoints

② Compute the forcing F_{ij} → advection of geostrophic relative + planetary vorticity by the geostrophic wind

③ Solve Poisson's eq $\nabla\chi_{ij} + F_{ij} = 0$ subject to const ϕ boundaries a N and S walls. We use successive over relaxation (SOR). We obtai geopotential tendency field χ_{ij}

④ Advance the ϕ_{ij} field using the computed tendency.
- ⓐ on first timestep use Euler scheme
- ⓑ on subsequent timesteps use leapfrog scheme
- ⓒ periodically interupt leapfrog with Euler-Backward for a time or two

⑤ Goto ② and repeat steps ②-④ until forecast has reached the fut time you desire.

⑥ Post process output
- ⓐ compute u_g, v_g
- ⓑ 〃 ζ_g
- ⓒ make contour map → 500 mb ϕ + vorticity
→ streamlines + isotachs.

* linear shallow water eq. in NWP

MET$\overset{5541}{?}$ (NWP): O'Brien (1 hour, 1995)

- A simple version of the linear shallow water equations for meteorology and oceanography is

$$① \begin{cases} u_t = fv - g'h_x \\ v_t = -fu - g'h_y \\ h_t = -H(u_x + v_y) \end{cases}$$

a) Write down a centered in time, centered in space finite difference approximation.

b) If $f = 0$, and $v = 0$; derive the linear von Neuman (CFL) stability equation.

c) If $f = 0$, derive the CFL Condition.

d) If $f \neq 0$, will it make the condition in (c) less stringent or more stable?

a)

$$U_{ij}^{n+1} = U_{ij}^{n-1} - 2\Delta t\, g'(h_{i+1,j}^{n} - h_{i-1,j}^{n})/2\Delta x + 2\Delta t f V_{ij}^{n}$$

$$V_{ij}^{n+1} = V_{ij}^{n-1} - 2\Delta t\, g'(h_{i,j+1}^{n} - h_{i,j-1}^{n})/2\Delta y - 2\Delta t f U_{ij}^{n}$$

$$h_{ij}^{n+1} = h_{ij}^{n-1} - 2\Delta t\, H(U_{i+1,j}^{n} - U_{i-1,j}^{n})/2\Delta x - 2\Delta t\, H(V_{i,j+1}^{n} - V_{i,j-1}^{n})/2\Delta y$$

Rewrite this system of eqs as

$$\left.\begin{aligned} U_{ij}^{n+1} &= U_{ij}^{n-1} - \left(\frac{g'\Delta t}{\Delta x}\right)(h_{i+1,j}^{n} - h_{i-1,j}^{n}) + 2\Delta t f V_{ij}^{n} \\ V_{ij}^{n+1} &= V_{ij}^{n-1} - \left(\frac{g'\Delta t}{\Delta y}\right)(h_{i,j+1}^{n} - h_{i,j-1}^{n}) - 2\Delta t f U_{ij}^{n} \\ h_{ij}^{n+1} &= h_{ij}^{n-1} - \left(\frac{H\Delta t}{\Delta x}\right)(U_{i+1,j}^{n} - U_{i-1,j}^{n}) - \left(\frac{H\Delta t}{\Delta y}\right)(V_{i,j+1}^{n} - V_{i,j-1}^{n}) \end{aligned}\right] ②$$

b) With $f = 0 = V$ our system of eqs ② reduces to

$$\left.\begin{aligned} U_{ij}^{n+1} &= U_{ij}^{n-1} - \left(\frac{g'\Delta t}{\Delta x}\right)(h_{i+1,j}^{n} - h_{i-1,j}^{n}) \\ 0 &= -\left(\frac{g'\Delta t}{\Delta y}\right)(h_{i,j+1}^{n} - h_{i,j-1}^{n}) \leftarrow \\ h_{ij}^{n+1} &= h_{ij}^{n-1} - \left(\frac{H\Delta t}{\Delta x}\right)(U_{i+1,j}^{n} - U_{i-1,j}^{n}) \end{aligned}\right] ③$$

diagnostic relation. Not part of stability analysis in this case. Thus we have no y dependence, only x and t.

We assume perturbations of the form

$$\begin{cases} U_{ij}^{n} = U^{s} e^{ikm\Delta x} \\ h_{ij}^{n} = h^{s} e^{ikm\Delta x} \end{cases}$$

∵ S is time index (n)
m is x axis index (i)
n is y 〃 〃 (j)

Substituting these perturbations into ③ yields (after cancelling common factor $e^{ikm\Delta x}$)

$$\begin{cases} U^{s+1} = U^{s-1} - \left(\frac{g'\Delta t}{\Delta x}\right) h^{s}\, 2i \sin k\Delta x \\ h^{s+1} = h^{s-1} - \left(\frac{H\Delta t}{\Delta x}\right) U^{s}\, 2i \sin k\Delta x \end{cases}$$

Rewrite this as a matrix eq

$$W^{s+1} = W^{s-1} + B W^{s}$$

where $W^{s} = \begin{pmatrix} U^{s} \\ h^{s} \end{pmatrix}$ And $B = \begin{pmatrix} 0 & -\left(\frac{g'\Delta t}{\Delta x}\right) 2i \sin k\Delta x \\ -\left(\frac{H\Delta t}{\Delta x}\right) 2i \sin k\Delta x & 0 \end{pmatrix}$

Define an amplification matrix G such that $W^{s+1} = GW^{s}$, $W^{s-1} = G^{-1}W^{s}$ then our matrix eq. above becomes

$$G = G^{-1} + B$$

or

$$G^{2} - BG - 1 = 0$$

The stability of the finite difference scheme (in the Von Neumann Sense) depends upon the spectral radius of the amplification matrix G. Denote spectral radius of G as $\sigma(G)$. The spectral radius of a matrix, such as (defined as the maximum eigenvalue of the matrix. Thus we need the root the above matrix eq. However, before computing these roots let us rewrite above quadratic in G as

$$G^{2} - 2iCG - 1 = 0$$

where $B = 2iC = 2i \underbrace{\begin{pmatrix} 0 & -\left(\frac{g'\Delta t}{\Delta x}\right) \sin k\Delta x \\ -\left(\frac{H\Delta t}{\Delta x}\right) \sin k\Delta x & 0 \end{pmatrix}}_{\to C \text{ matrix.}}$

So, given $G^{2} - 2iCG - 1 = 0$

the roots are

$$G = iC \pm \sqrt{-C^{2} + 1} = iC \pm (1 - C^{2})^{1/2}$$

Note that if the matrix norm $\|C\| < 1$ then $\sigma(G) = C^{2} + (1 - C^{2}) = 1$. That is, the scheme is stable if $1 - C^{2} > 0$ or equivalently $\|C\| < 1$.

We must determine the eigenvalues of the matrix C to see what conditio to requirement $\|C\| < 1$ places on parameters in the problem. The eige of the C matrix are roots of the characteristic eq. below.

$$\begin{vmatrix} -\lambda & -\left(\frac{g'\Delta t}{\Delta x}\right) \sin k\Delta x \\ -\left(\frac{H\Delta t}{\Delta x}\right) \sin k\Delta x & \underset{\llcorner \text{ eigenvalue.}}{-\lambda} \end{vmatrix} = 0$$

or

$$\lambda^{2} - (g'H)\left(\frac{\Delta t}{\Delta x}\right)^{2} \sin^{2} k\Delta x = 0$$

$$\lambda = \pm\sqrt{g'H}\left(\frac{\Delta t}{\Delta x}\right) \sin k\Delta x$$

As $\sin k\Delta x \in [-1, 1]$, the maximum eigenvalue, is

$$\lambda_{max} = \sqrt{g'H}\left(\frac{\Delta t}{\Delta x}\right)$$

Thus, the condition $\|C\| < 1$ implies

$$\lambda_{max} < 1 \quad \text{or} \quad \sqrt{g'H}\frac{\Delta t}{\Delta x} < 1$$

This is the "normal" CFL condition when we note that with $f = 0$ the sy of eqs only describes shallow water gravity waves. These waves have a p speed of $C = \sqrt{g'H}$. The CFL condition states that the wave can not tra a distance greater then Δx in time Δt.

c)

The system of pertinent eqs is

$$\left.\begin{aligned} u_{ij}^{n+1} &= u_{ij}^{n-1} - \left(\frac{g'\Delta t}{\Delta x}\right)(h_{i+1,j}^{n} - h_{i-1,j}^{n}) \\ v_{ij}^{n+1} &= v_{ij}^{n-1} - \left(\frac{g'\Delta t}{\Delta y}\right)(h_{i,j+1}^{n} - h_{i,j-1}^{n}) \\ h_{ij}^{n+1} &= h_{ij}^{n-1} - \left(\frac{H\Delta t}{\Delta x}\right)(u_{i+1,j}^{n} - u_{i-1,j}^{n}) - \left(\frac{H\Delta t}{\Delta y}\right)(v_{i+1,j}^{n} - v_{i-1,j}^{n}) \end{aligned}\right] ④$$

We have two spatial dimensions in this problem (x + y). Assume perturbations of the form

$$\left(\begin{aligned} u_{ij}^{n} &= u^{s} e^{i(km\Delta x + ln\Delta y)} \\ v_{ij}^{n} &= v^{s} e^{i(km\Delta x + ln\Delta y)} \\ h_{ij}^{n} &= h^{s} e^{i(kn\Delta x + ln\Delta y)} \end{aligned}\right.$$

Substitute these into ④

$$\left[\begin{aligned} u^{s+1} &= u^{s-1} - \left(\frac{g'\Delta t}{\Delta x}\right) h^{s}\, 2i\,\Omega{:}k\Delta x \\ v^{s+1} &= v^{s-1} - \left(\frac{g'\Delta t}{\Delta x}\right) h^{s}\, 2i\,\Omega{:}l\Delta y \\ h^{s+1} &= h^{s-1} - \left(\frac{H\Delta t}{\Delta x}\right) u^{s}\, 2i\,\Omega{:}k\Delta x - \left(\frac{H\Delta t}{\Delta y}\right) v^{s}\, 2i\,\Omega{:}l\Delta y \end{aligned}\right.$$

Rewrite as a matrix eq.

$$W^{s+1} = W^{s-1} + B W^{s}$$

where

$$W^{s} = \begin{pmatrix} u^{s} \\ v^{s} \\ h^{s} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & -\frac{g'\Delta t}{\Delta x} 2i\,\Omega{:}k\Delta x \\ 0 & 0 & -\frac{g'\Delta t}{\Delta y} 2i\,\Omega{:}l\Delta y \\ -\left(\frac{H\Delta t}{\Delta x}\right) 2i\,\Omega{:}k\Delta x & -\frac{H\Delta t}{\Delta y} 2i\,\Omega{:}l\Delta y & 0 \end{pmatrix}$$

Again define an amplification matrix G such that $W^{s+1} = G W^{s}$ and $W^{s-1} = G^{-1} W^{s}$ so that our matrix eq may be rewritten as

$$G = G^{-1} + B$$

or

$$G^{2} - BG - 1 = 0$$

or

$$G^{2} - 2iCG - 1 = 0$$

where C is matrix B but without the common factor $2i$. As before we find

$$G = iC \pm (1 - C^{2})^{1/2}$$

and so $\nu(G) = 1$ if $\|C\| < 1$. We must determine the eigenvalues of C. They are the roots of the characteristic eq.

$$\begin{vmatrix} -\lambda & 0 & -\frac{g'\Delta t}{\Delta x}\Omega{:}h\Delta x \\ 0 & -\lambda & -\frac{g'\Delta t}{\Delta y}\Omega{:}l\Delta x \\ -\left(\frac{H\Delta t}{\Delta x}\right)\Omega{:}k\Delta x & -\frac{H\Delta t}{\Delta y}\Omega{:}l\Delta y & -\lambda \end{vmatrix} = 0$$

$$-\lambda^{3} + \lambda\left[g'H\left(\frac{\Delta t}{\Delta y}\right)^{2}\Omega{:}^{2}l\Delta y + g'H\left(\frac{\Delta t}{\Delta x}\right)^{2}\Omega{:}^{2}k\Delta x\right] = 0$$

$$\lambda(-\lambda^{2} + [\quad]) = 0$$

$$\lambda = 0 \text{ or } \lambda^{2} = (g'H)\left(\left(\frac{\Delta t}{\Delta x}\right)^{2}\Omega{:}^{2}k\Delta x + \left(\frac{\Delta t}{\Delta y}\right)^{2}\Omega{:}^{2}l\Delta y\right)$$

$$\text{Note: } \lambda_{max} = \sqrt{g'H}\left[\left(\frac{\Delta t}{\Delta x}\right)^{2}\Omega{:}^{2}k\Delta x + \left(\frac{\Delta t}{\Delta y}\right)^{2}\Omega{:}^{2}l\Delta y\right]^{1/2}$$

The CFL condition for stability is $\lambda_{max} \le 1$. Since $\Omega{:}^{2}k\Delta x$ and $\Omega{:}^{2}l\Delta y$ range from 0 to 1 inclusive, the CFL condition is

$$\lambda_{max} = \sqrt{g'H}\left(\left(\frac{\Delta t}{\Delta x}\right)^{2} + \left(\frac{\Delta t}{\Delta y}\right)^{2}\right)^{1/2} \le 1$$

Note that if $\Delta x = \Delta y = d$

$$\frac{(\Delta t)^{2}}{d^{2}} + \frac{(\Delta t)^{2}}{d^{2}} = \frac{2(\Delta t)^{2}}{d^{2}} \text{ and } \left(\frac{2(\Delta t)^{2}}{d^{2}}\right)^{1/2} = \sqrt{2}\frac{\Delta t}{d}$$

So $\lambda_{max} = \sqrt{g'H}\,\frac{\sqrt{2}\Delta t}{d} \le 1$

$$\sqrt{g'H}\,\frac{\Delta t}{d} \le \frac{1}{\sqrt{2}} \sim 0.707$$

$\uparrow$

more stringent CFL condition in two dimensions.

d)

We substitute the u, v, h perturbation from part (c) into the system of eqs we obtain a matrix eq for the amplification factor, G,

$$G^{2} - BG - 1 = 0$$

where

$$B = \begin{pmatrix} 0 & 2\Delta t f & -\left(\frac{g'\Delta t}{\Delta x}\right) 2i\,\Omega{:}k\Delta x \\ -2\Delta t f & 0 & -\left(\frac{g'\Delta t}{\Delta y}\right) 2i\,\Omega{:}l\Delta y \\ -\frac{H\Delta t}{\Delta x}\Omega{:}k\Delta x\, 2i & 2i\left(-\frac{H\Delta t}{\Delta y}\right)\Omega{:}l\Delta y & 0 \end{pmatrix}$$

Define a matrix C such that the above quadratic in G becomes

$$G^{2} - 2iCG - 1 = 0$$

$$C = \begin{pmatrix} 0 & -i\Delta t f & -\left(\frac{g'\Delta t}{\Delta x}\right)\Omega{:}k\Delta x \\ i\Delta t f & 0 & -\left(\frac{g'\Delta t}{\Delta y}\right)\Omega{:}l\Delta y \\ -\frac{H\Delta t}{\Delta x}\Omega{:}k\Delta x & -\frac{H\Delta t}{\Delta y}\Omega{:}l\Delta y & 0 \end{pmatrix}$$

The eigenvalues of this matrix are roots of the characteristic eq. $|C - I\lambda| =$

or

$$-\lambda^{3} + \lambda\left[g'H\left(\frac{\Delta t}{\Delta y}\right)^{2}\Omega{:}^{2}l\Delta y - i^{2}(f\Delta t)^{2} + g'H\left(\frac{\Delta t}{\Delta x}\right)^{2}\Omega{:}^{2}k\Delta x\right]$$

$$\cancel{-i\Delta t f g'H\frac{\Delta t}{\Delta x}\frac{\Delta t}{\Delta y}\Omega{:}l\Delta y\,\Omega{:}k\Delta x} + \cancel{i\Delta t f g'H\frac{\Delta t}{\Delta x}\frac{\Delta t}{\Delta y}\Omega{:}l\Delta y\,\Omega{:}k\Delta x} = 0$$

$$\lambda\left(-\lambda^{2} + g'H\left(\frac{\Delta t^{2}}{\Delta x^{2}}\Omega{:}^{2}k\Delta x + \frac{\Delta t^{2}}{\Delta y^{2}}\Omega{:}^{2}l\Delta y\right) + f^{2}\Delta t^{2}\right) = 0$$

$$\lambda = 0 \text{ or } \lambda^{2} = g'H\left[\left(\frac{\Delta t}{\Delta x}\right)^{2}\Omega{:}^{2}k\Delta x + \left(\frac{\Delta t}{\Delta y}\right)^{2}\Omega{:}^{2}l\Delta y\right] + (f\Delta t)^{2}$$

The max eigenvalue is

$$\lambda_{max} = \left[g'H\left(\left(\frac{\Delta t}{\Delta x}\right)^{2} + \left(\frac{\Delta t}{\Delta y}\right)^{2}\right) + (f\Delta t)^{2}\right]^{1/2}$$

The CFL condition is

$$\left[g'H\left(\left(\frac{\Delta t}{\Delta x}\right)^{2} + \left(\frac{\Delta t}{\Delta y}\right)^{2}\right) + (f\Delta t)^{2}\right]^{1/2} \le 1$$

Note that

$$g'H\left(\left(\frac{\Delta t}{\Delta x}\right)^{2} + \left(\frac{\Delta t}{\Delta y}\right)^{2}\right) \le 1 - (f\Delta t)^{2}$$

which is a <u>more stringent</u> CFL condition than when $f = 0$ and the conditio

$$g'H\left(\left(\frac{\Delta t}{\Delta x}\right)^{2} + \left(\frac{\Delta t}{\Delta y}\right)^{2}\right) \le 1$$

Thus, the presence of $f \ne 0$ makes the scheme less stable (or equivalently, the CFL condition more stringent).

* linear shallow water eq in NWP + stability analysis

5541

MET? (NWP): O'Brien (1 hour) *

- The linear shallow water equation in 1-dimension are

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = fv - g\frac{\partial h}{\partial x} \qquad (1)$$

$$\frac{\partial v}{\partial t} + A\frac{\partial v}{\partial x} = -fu \qquad (2)$$

$$\frac{\partial h}{\partial t} + A\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0 \qquad (3)$$

where f, A, H, g are constant parameters.

A. Write down an explicit centered in time, centered in space finite difference approximation for these equations. For example you might use the Arakawa C-scheme.

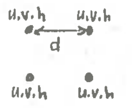
B. For $f = 0$, determine the C-F-L linear stability criteria for your equations in A.

C. For $A = 0$, determine the C-F-L linear stability criteria for equations in A.

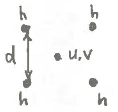
D. IF A and f are not zero, is it possible to calculate the linear stability criteria analytically?

* look at the previous question

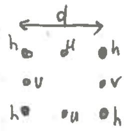
Sol) In the shallow water eqs we have mass variables (h) and wind variables (u,v). In casting the PDEs (1)–(3) which constitute the shallow water system the question arises as to how one places mass + wind variables on the finite difference grid. Below are 5 possible arrangements of the dependent variables



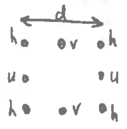
A
(nonstaggered)



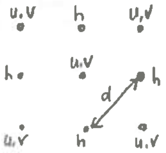
B



C



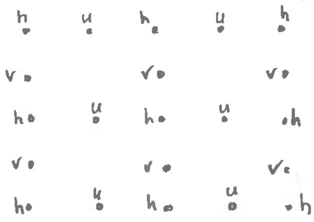
D



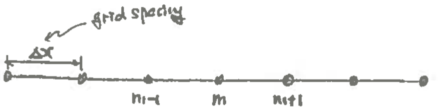
E ← rotate B grid by $\pi/4$

When computing horizontal derivatives by means of finite differences at some gridpoint much of the data comes from adjacent points. This suggests that it would be more efficient if the variables are staggered in the horizontal. Staggered grids can reduce the problem of solution separation that occurs with leapfrog time differencing on a nonstaggered grid. Many models use the B or C grid.

We will use the C grid upon which we will finite difference (1)–(3)



We have a problem. We started this problem by stating that we are dealing with the linear SWE in 1-D. We need 2D to define a C grid. So we will drop the C grid and use the A grid. In the A grid every grid point (i,j) carries all variables ⇒ h, u, v



Let m be the spatial index (in the x direction) ; Δt is timestep
n be the temporal index

Using explicitly centered in time and centered in space finite differences approx. (1)–(3) at point m, time n as

$$\begin{cases} \dfrac{U_m^{n+1} - U_m^{n-1}}{2\Delta t} + A\left(\dfrac{U_{m+1}^n - U_{m-1}^n}{2\Delta x}\right) = fV_m^n - g\left(\dfrac{h_{m+1}^n - h_{m-1}^n}{2\Delta x}\right) & (4) \\ \dfrac{V_m^{n+1} - V_m^{n-1}}{2\Delta t} + A\left(\dfrac{V_{m+1}^n - V_{m-1}^n}{2\Delta x}\right) = -fU_m^n & (5) \\ \dfrac{h_m^{n+1} - h_m^{n-1}}{2\Delta t} + A\left(\dfrac{h_{m+1}^n - h_{m-1}^n}{2\Delta x}\right) + H\left(\dfrac{U_{n+1}^n - U_{m-1}^n}{2\Delta x}\right) = 0 & (6) \end{cases}$$

- Consider the 1D problem ($f=0$)

$$\begin{cases} \dfrac{\partial u}{\partial t} = -A\dfrac{\partial u}{\partial x} - g\dfrac{\partial h}{\partial x} \\ \dfrac{\partial v}{\partial t} = -A\dfrac{\partial v}{\partial x} \\ \dfrac{\partial h}{\partial t} = -A\dfrac{\partial h}{\partial x} - H\dfrac{\partial u}{\partial x} \end{cases}$$

Finite difference these PDE's using centered in space, centered in time appr

$$\begin{cases} u_j^{n+1} = u_j^{n-1} - \dfrac{A\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n) - \dfrac{g\Delta t}{\Delta x}(h_{j+1}^n - h_{j-1}^n) \\ V_j^{n+1} = V_j^{n-1} - \dfrac{A\Delta t}{\Delta x}(V_{j+1}^n - V_{j-1}^n) \\ h_j^{n+1} = h_j^{n-1} - \dfrac{A\Delta t}{\Delta x}(h_{j+1}^n - h_{j-1}^n) - \dfrac{H\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n) \end{cases}$$

Define

$$\begin{cases} u_j^n = U_n e^{ikj\Delta x} \\ V_j^n = V_n e^{ikj\Delta x} \\ h_j^n = h_n e^{ikj\Delta x} \end{cases} \qquad \lambda = \frac{\Delta t}{\Delta x} \qquad \alpha = k\Delta x$$

Note

$$e^{ik\Delta x} - e^{-ik\Delta x} = 2i\sin k\Delta x$$

We substitute these definitions into our finite difference eqs above

$$\begin{cases} U_{n+1} = U_{n-1} - A\lambda U_n 2i\sin\alpha - g\lambda h_n 2i\sin\alpha \\ V_{n+1} = V_{n-1} - A\lambda V_n 2i\sin\alpha \\ h_{n+1} = h_{n-1} - A\lambda h_n 2i\sin\alpha - H\lambda U_n 2i\sin\alpha \end{cases}$$

We want to obtain a single eq in U. Toward this end form an eq for and U_n from the U_{n+1} eq above.

$$\begin{cases} U_{n+2} - U_n = -A\lambda U_{n+1} 2i\sin\alpha - g\lambda h_{n+1} 2i\sin\alpha \\ U_n - U_{n-2} = -A\lambda U_{n-1} 2i\sin\alpha - g\lambda h_{n-1} 2i\sin\alpha \end{cases}$$

Subtract the U_n eq from the U_{n+2} eq.

$$U_{n+2} - U_n - U_n + U_{n-2} = -A\lambda 2i\sin\alpha U_{n+1} + A\lambda 2i\sin\alpha U_{n-1}$$
$$- 2ig\lambda\sin\alpha(\underline{h_{n+1} - h_{n-1}})$$

$$U_{n+2} - 2U_n + U_{n-2} = 2iA\lambda\sin\alpha(U_{n-1} - U_{n+1}) - 2ig\lambda\sin\alpha(-2iA\lambda\sin\alpha h_n)$$
$$- 2ig\lambda\sin\alpha(-2iH\lambda\sin\alpha U_n)$$

or

$$U_{n+2} - 2U_n + U_{n-2} = 2iA\lambda\sin\alpha(U_{n-1} - U_{n+1}) - 4gH\lambda^2\sin^2\alpha U_n$$
$$- 2iA\lambda\sin\alpha(\underline{-2ig\lambda\sin\alpha h_n})$$

$$U_{n+2} + U_{n-2} = 2iA\lambda\sin\alpha(U_{n-1} - U_{n+1}) - (4gH\lambda^2\sin^2\alpha - 2)U_n$$
$$- 2iA\lambda\sin\alpha(U_{n+1} - U_{n-1} + 2iA\lambda\sin\alpha U_n)$$

$$U_{n+2} + U_{n-2} = -2iA\lambda\sin\alpha(-U_{n-1} + U_{n+1}) - (4gH\lambda^2\sin^2\alpha - 2 - 4A^2\lambda^2\sin^2\alpha)U_n$$
$$- 2iA\lambda\sin\alpha(-U_{n-1} + U_{n+1})$$

$$U_{n+2} + U_{n-2} = 4iA\lambda\sin\alpha(U_{n-1} - U_{n+1}) - (4gH\lambda^2\sin^2\alpha - 2 - 4A^2\lambda^2\sin^2\alpha)U_n$$

We define an amplification factor G such that

$$U_{n+1} = GU_n$$

Then

$$G^2 + G^{-2} = 4iA\lambda\sin\alpha(G^{-1} - G) - (4gH\lambda^2\sin^2\alpha - 2 - 4A^2\lambda^2\sin^2\alpha)$$

Multiply thru by G^2

$$G^4 + 4iA\lambda\sin\alpha G^3 + (4gH\lambda^2\sin^2\alpha - 2 - 4A^2\lambda^2\sin^2\alpha)G^2 - 4iA\lambda\sin\alpha G + 1 = 0$$

$$\vdots$$

Stuck. How do I solve for G??

- We consider the 1D problem $(A=0)$

$$\begin{cases} \frac{\partial u}{\partial t} = fv - g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} = -fu \\ \frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x} \end{cases}$$

we finite difference this using centered in time + centered in space schemes

we define

$$\begin{cases} U_j^n = U_n e^{ikj\Delta x} \\ V_j^n = V_n e^{ikj\Delta x} \\ h_j^n = h_n e^{ikj\Delta x} \end{cases} \qquad \alpha = k\Delta x, \ \lambda = \frac{\Delta t}{\Delta x}, \qquad \beta = f\Delta t$$

Before we use these, finite difference the PDEs

$$\begin{cases} u_j^{n+1} = u_j^{n-1} + 2\Delta t f V_j^n - \frac{g\Delta t}{\Delta x}(h_{j+1}^n - h_{j-1}^n) \\ V_j^{n+1} = V_j^{n-1} - 2\Delta t f u_j^n \\ h_j^{n+1} = h_j^{n-1} - \frac{H\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n) \end{cases}$$

Make use of α, β, λ definitions. Substitute for U_j^n, V_j^n, h_j^n

$$\begin{cases} U_{n+1} = U_{n-1} + 2\beta V_n - g\lambda h_n 2i\sin\alpha \\ V_{n+1} = V_{n-1} - 2\beta U_n \\ h_{n+1} = h_{n-1} - H\lambda U_n 2i\sin\alpha \end{cases}$$

Rewrite this as

$$\begin{cases} U_{n+1} - U_{n-1} = 2\beta V_n - 2ig\lambda\sin\alpha h_n \\ V_{n+1} - V_{n-1} = -2\beta U_n \\ h_{n+1} - h_{n-1} = -2iH\lambda\sin\alpha U_n \end{cases}$$

We want to obtain an eq. in U_n only

If we rewrite the U_{n+1} eq for U_{n+2} and U_n

$$\begin{cases} U_{n+2} - U_n = 2\beta V_{n+1} - 2ig\lambda\sin\alpha h_{n+1} \\ U_n - U_{n-2} = 2\beta V_{n-1} - 2ig\lambda\sin\alpha h_{n-1} \end{cases}$$

Subtract the lower from the upper

$$U_{n+2} - U_n - U_n + U_{n-2} = 2\beta(\underline{V_{n+1} - V_{n-1}}) - 2ig\lambda\sin\alpha(\underline{h_{n+1} - h_{n-1}})$$

$$U_{n+2} - 2U_n + U_{n-2} = 2\beta(-2\beta U_n) - 2ig\lambda\sin\alpha(-2iH\lambda\sin\alpha)U_n$$

$$U_{n+2} - 2U_n + U_{n-2} = -4\beta^2 U_n - 4gH\lambda^2\sin^2\alpha U_n$$

If an amplification factor G exists such that

$$\underline{U_{n+2} = GU_n}$$

then upon substitution the above eq. in U_n becomes

$$G - 2 + G^{-1} = -4\beta^2 - 4gH\lambda^2\sin^2\alpha$$

or

$$G^2 + (-2 + 4\beta^2 + 4gH\lambda^2\sin^2\alpha)G + 1 = 0$$

Solve for G.

$$G = -(-1 + 2\beta^2 + 2gH\lambda^2\sin^2\alpha) \pm \sqrt{(-1 + 2\beta^2 + 2gH\lambda^2\sin^2\alpha)^2 - 1}$$
$$= (1 - 2\beta^2 - 2gH\lambda^2\sin^2\alpha) \pm i[1 - [1 - 2\beta^2 - 2gH\lambda^2\sin^2\alpha]^2]^{1/2}$$

If the radical is real, then $|G| = 1$ for all α. This requires

$$[1 - 2\beta^2 - 2gH\lambda^2\sin^2\alpha]^2 \le 1$$
$$-1 \le 1 - 2\beta^2 - 2gH\lambda^2\sin^2\alpha \le 1$$
$$-2 \le -2\beta^2 - 2gH\lambda^2\sin^2\alpha \le 0$$
$$\boxed{1 \ge \beta^2 + gH\lambda^2\sin^2\alpha \ge 0} \longrightarrow$$

$$0 \le gH\lambda^2\sin^2\alpha \le 1 - \beta^2$$
$$0 \le \sqrt{gH}\lambda\sin\alpha \le (1-\beta)^{1/2}$$
$$(\sqrt{gH})\frac{\Delta t}{\Delta x}\underset{\hookrightarrow \text{ this is } \le 1}{\sin\alpha} \le (1-\beta^2)^{1/2}$$

so $\sqrt{gH}\frac{\Delta t}{\Delta x} \le (1-\beta^2)^{1/2}$

For this to be true

$$\beta^2 \le 1$$

$$\underline{|f\Delta t| \le 1 \text{ and } \frac{\sqrt{gH}\Delta t}{\Delta x} \le (1-\beta^2)^{1/2}}$$

We note that the Corrolis term $f\Delta t$ reduces CFL condition below unity. In practice the timestep is a small fraction of the inertial period $\frac{f}{2\pi}$ so that this is not a serious problem.

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note

The matrix A whose element is a_{jk}

matrix norms

- maximum (or infinity) norm

$$\|A\|_1 = \underset{k}{\text{maximum}} \sum_{j=1}^{n} |a_{jk}| \quad : \text{max column sum}$$

$$\|A\|_\infty = \max_{j} \sum_{k=1}^{n} |a_{jk}| \quad : \text{max row sum.}$$

$\sigma(A) = |A|$: spectral radius of matrix A, is defined as the magnitude of maximum eigenvalue

$$\sigma(A) < \|A\|_1$$

vector norm

- Euclidean (or L_2) norm : the length of a vector

$$\|X\|_2 = |X| = \sqrt{\sum_{j=1}^{n} |X_j|^2}$$

if matrix X is given by $X^T = [X_1 \ X_2 \ X_3 \ \cdots \ X_n]$

- L_p norm

$$\|X\| = \sqrt[p]{\sum_{j=1}^{n} |X_j|^p}$$

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 6 \end{bmatrix}$ $\quad \|A\|_1 = 3+4+6 = 13$, $\quad \|A\|_\infty = 0+0+6 = 6$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -2-\lambda & 4 \\ 0 & 0 & 6-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda)(6-\lambda) = 0$$

$\sigma(A) = 6 \qquad 6 < 13$

✱ more sol + note

$$\begin{cases} u_t + Au_x = fv - gh_x \\ v_t + Av_x = -fu - gh_y \\ h_t + Ah_x = -Hu_x \end{cases}$$

f=0

$$\begin{cases} u_t + Au_x + gh_x = 0 \\ h_t + Ah_x + Hu_x = 0 \end{cases}$$

Leapfrog + centered difference

$$\overline{u_t}^t + \lambda \overline{u_x}^x + g'\overline{h_x}^x = 0 \qquad \lambda = \frac{A\Delta t}{\Delta x}$$

$$\overline{h_t}^t + \lambda \overline{h_x}^x + H'\overline{u_x}^x = 0 \qquad g' = g\frac{\Delta t}{\Delta x}, \quad H' = H\frac{\Delta t}{\Delta x}$$

$$(u,h) = [U_s, h_s]e^{ijk\Delta x}$$

$$U_{s+1} - U_{s-1} + \lambda(2i\sin k\Delta x)U_s + g'(2i\sin k\Delta x)h_s = 0$$

$$h_{s+1} - h_{s-1} + \lambda(2i\sin k\Delta x)h_s + H'(2i\sin k\Delta x)U_s = 0$$

$$W_{s+1} = \begin{bmatrix} U_{s+1} \\ h_{s+1} \end{bmatrix}$$

$$W_{s+1} = W_{s-1} + BW_s \qquad \because B = \begin{bmatrix} 2i\lambda\sin k\Delta x & 2ig'\sin k\Delta x \\ H'(2i\sin k\Delta x) & 2i\lambda\sin k\Delta x \end{bmatrix}$$

$$W_{s+1} = GW_s \ ; \ W_s = GW_{s-1}$$

$$G = G^{-1} + B$$

$$\underline{G^2 - BG - 1 = 0} \longleftarrow \text{matrix eq.}$$

$$\because B = 2iC$$

$$G^2 - 2iCG - 1 = 0$$

$$G = iC \pm [-C^2+1]^{1/2}$$

if $\underset{\uparrow \text{matrix norm}}{\|C\|} < 1$, then $|G| = 1$ ⟶ spectral radius of matrix G.

$$1 - C^2 > 0$$

$$\|C\| < 1$$

$$C = \begin{bmatrix} \lambda\sin k\Delta x & g'\sin k\Delta x \\ H'\sin k\Delta x & \lambda\sin k\Delta x \end{bmatrix}$$

Roots are Λ

$$(\lambda\sin k\Delta x - \Lambda)^2 - g'H'\sin^2 k\Delta x = 0$$

$$\lambda\sin k\Delta x - \Lambda = \pm\sqrt{g'H'}\sin k\Delta x$$

$$\Lambda = (\lambda - \sqrt{gH'})\sin k\Delta x$$

$$\boxed{(\Lambda)_{max} = \lambda + \sqrt{g'H'}} = \frac{A\Delta t}{\Delta x} + \sqrt{gH}\frac{\Delta t}{\Delta x}$$

A=0

$$\begin{cases} u_t = fv - gh_x \\ v_t = -fu \\ h_t = -Hu_x \end{cases}$$

$$\begin{cases} \overline{u_t}^t = 2f'v - g'\overline{h_x}^x \\ \overline{v_t}^t = -2f'u \\ \overline{h_t}^t = -H'\overline{u_x}^x \end{cases} \qquad \because \begin{cases} f' = f\Delta t \\ g' = g\frac{\Delta t}{\Delta x} \\ H' = H\frac{\Delta t}{\Delta x} \end{cases}$$

Let $[u, v, h] = [U_s, V_s, h_s]e^{ijk\Delta x}$

$$U_{s+1} = U_{s-1} + f'V_s - g'2i\sin k\Delta x\, h_s$$

$$V_{s+1} = V_{s-1} - f'U_s$$

$$h_{s+1} = h_{s-1} - H'2i\sin k\Delta x\, U_s$$

$$W_s = \begin{bmatrix} U_s \\ V_s \\ h_s \end{bmatrix} \rightarrow W_{s+1} = W_{s-1} + 2BW_s$$

where $B = \begin{bmatrix} 0 & +f' & -g'i\sin k\Delta x \\ -f' & 0 & 0 \\ -H'2i\sin k\Delta x & 0 & 0 \end{bmatrix}$

$$W_{s+1} = GW_s \ ; \ W_s = GW_{s-1}$$

$$G = G^{-1} + 2B$$

$$G^2 - 2BG - 1 = 0 \longrightarrow G = B \pm (B^2+1)^{1/2}$$

Eigenvalues of B are

$$B = \begin{pmatrix} 0-\lambda & +f' & -ig'\sin k\Delta x \\ -f' & 0-\lambda & 0 \\ -H'i\sin k\Delta x & 0 & 0-\lambda \end{pmatrix}$$

$$-\lambda[\lambda^2 + f_1^2] + g'H'\sin^2 k\Delta x\,\lambda = 0$$

$$\lambda^2 + f_1^2 = g'H'\sin^2 k\Delta x$$

$$\boxed{\lambda^2 = g'H'\sin^2 k\Delta x - f_0'^2}$$

MET? (NWP plus Advanced Dynamics): O'Brien (1 hour, 1995)

- You wish to solve the Quasi-Geostrophic equations in a rectangular beta-channel. Assume the channel length, $x \in [0,1]$ and width, $y \in [a,b]$.

a) Write down a set of equations for this problem.
   (1) Define the dependent and independent variables.
   (2) Specify appropriate boundary conditions.

b) Outline in some detail how to use a spectral method in x to reduce the streamfunction equation to a set of ordinary 3 point difference equations.
   (1) Define how to solve these difference equations.

4. O'Brien / Numerical Analysis / 60 minutes

A. Define an iterative process

$$X^{(m+1)} = BX^{(m)} + C$$

where X and C are vectors and B is a square matrix. Show that the process converges if $||B|| < 1$.

B. SOR is defined

$$X^{(m+1)} = X^{(m)} + \omega R$$

where

$$R = D^{-1}b - X^{(m)} + LX^{(m+1)} + UX^{(m)}$$

where X, R, b are vectors, D ia a diagonal matrix, L is a strictly lower triangular matrix and U is an upper triangular matirx.

1) derive the interation matrix for SOR

2) If the interation matrix is convergent, show that the process converges to the solution of the problem

$$(I - L - U)X = D^{-1}b$$

C. Define the terms in terms of the oscillation equation, $\frac{\partial F}{\partial t} = + i\omega F$

Explicit method
Implicit method
Semi-implicit method

D. Consider the inertial oscillation problem

$$\frac{\partial u}{\partial t} = fv$$

$$\frac{\partial v}{\partial t} = - fu$$

1) show that forward difference is unstable and that an implicit method is stable.

Hint: let $w = u + iv$.

1. S.O.R. (1 Hour) (OBrien
Suppose you wanted to solve

$$\frac{\partial}{\partial x} K(x,y) \frac{\partial}{\partial x} Q + \frac{\partial}{\partial y} K(x,y) \frac{\partial}{\partial y} Q - H(x,y)\ Q - F(x,y) = 0$$

in a rectangle $x \in (0,L_x)$, $y \in (0,L_y)$ with boundary conditions $Q(x,y) = 0$ an boundary using SOR. $H(x,y) > 0$; $K(x,y) > 0$ for all x,y.

Suppose that there are J+1 grid points in x and I+1 grid points in y.

Write down finite difference approximations that are second order in x and y. Carefully describe the iterative procedure called successive overrelaxation for this problem. How would you estimate an approximation to the optimum relaxation coefficient?

Do you know apriori that the problem will converge? What is the solution if F=0.

Show that SOR is equilvalent to solving for the steady solution of the above problem with $\alpha \frac{\partial Q}{\partial t}$ on the right hand side. Find α in terms of the problem.

JJO:rkk

O'B[illegible]en
4[illegible]in.

13. Starr ~~[illegible]~~ published a book on Negative Eddy Vis~~cosity which~~ shows that many fluid measurements indicate counter-gradient heat transfer or momentum transfer, or, the idea of a negative diffusivity or viscosity.

a) What do we mean by counter-gradient flow?

b) Write the Euler-form of the primitive equations of motion for an incompressible, two-dimensional fluid and derive an expression which proves that the molecular viscosity must be positive. (Hint, derive a total energy equation)

c) What is meant by an eddy viscosity or diffusivity? What are its units?

d) The formal derivation of the Eulerian form of the Navier-Stokes equations doesn't contain any terms with eddy viscosity. Derive an averaged form of the equations which are not closed and require the introduction of an eddy viscosity for closure.

e) Show that the average eddy viscosity in a fluid is expected to be positive.

Dr. James O'Brien

1 hour

Ph.D. Question for John Schultz

5. This question involves a partial description of the GFDL model.

   a) What are the dependent and independent variables in the GFDL Ocean Model?

   b) What are all the boundary conditions needed to specify a solution to a problem?

   c) What are the adjustable parameters in this model?

   d) What range of initial conditions can be used for this model?

   e) How does this model handle convective overturning?

rien
min.

1. Starr published a book on Negative Eddy Viscosity which shows that many fluid measurements indicate counter-gradient heat transfer or momentum transfer, or, the idea of a negative diffusivity or viscosity.

   a) What do we mean by counter-gradient flow?

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   e) Show that the average eddy viscosity in a fluid is expected to be positive.

2. Consider the linearized equations of motion

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = fv \qquad (1)$$

$$\frac{\partial v}{\partial t} + A\frac{\partial v}{\partial x} = -fu \qquad (2)$$

for which the following numerical scheme is suggested

$$u_j^{n+1} = u_j^n - \frac{A\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) - \Delta tf\, v^{n+1} \qquad (3)$$

$$v_j^{n+1} = v_j^n - \frac{A\Delta t}{2\Delta x}(v_{j+1}^n - v_{j-1}^n) + \Delta tf\, u^{n+1} \qquad (4)$$

   (a) Use the von Neumann stability test to determine under what conditions this scheme is stable.

   (b) Please suggest a finite difference scheme which is stable for all Δt, Δx.

* energy balance at sea sfc.

OCP5551

MET? (Air-Sea): O'Brien (1 hour)

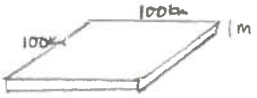
✓

- Suppose you desired to determine the reason for the increase in temperature at the sea surface over a month.

a) What energy balance equation would you use?

b) What measurements would be required to measure all the energy fluxes? Consider that you are concerned with the average sea-surface temperature in a 100 by 100 km horizontal slab of thickness, 1m.

(You really need to write a short essay to answer this question.)



?

refer to Nicholson's question

* Prandtl layer → log-wind profile

└ = sfc layer = const flux layer

✓ MET? : O'Brien (30 minutes)

- In the atmospheric Prandtl layer, the surface stress is considered to be independent of height. This allows us to define a non-dimensional wind shear, S,

$$✓\quad S = \frac{kz}{U_*}\frac{\partial V}{\partial z} \qquad (1)$$

where V is the average horizontal speed. In adiabatic conditions, S is unity and we can integrate S to obtain the log wind profile.

(a) Integrate S to obtain the log profile and interpret physically the constant of integration.

(b) For adiabatic conditions, what is the variation with height of the eddy viscosity, K_M, implied by (1)?

(c) How does $V(z)$ deviate from a log profile during stable and unstable conditions? A diagram might help.

(a)

Assume adiabatic conditions + integrate S in the vertical.

$$1 = \frac{kz}{u_*}\frac{\partial V}{\partial z}$$

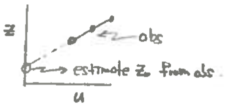
$$\int_{z_0}^{z}\frac{u_*}{kz}dz = \int_{z_0}^{z}\frac{\partial V}{\partial z}dz$$

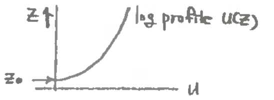
$$\frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) = V(z) - V(z_0)$$

So

$$V(z) = V(z_0) + \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right)$$

∵ z_0 is the level above the sfc (but below z) where the wind $V(z_0)$ is zero.





We call z_0 the (aerodynamic) roughness length. Generally $z_0 < z$ is chosen so that $V(z_0) = 0$. The roughness length z_0 varies widely depending on the physical characteristics of the sfc. Typical values of z_0 range from 10^{-5} m over smooth ice to 10^{1} m over large buildings. Although z_0 is NOT equal to the height of the individual roughness elements on the ground there is a 1 to 1 correspondence between those roughness elements and their particular z_0. Once z_0 is computed for a particular sfc, it does not vary with wind speed, stability, or stress. It can change if the roughness elements on the sfc change → a forrested area cleared of its trees.

z_{0f} differs from z_{0c}

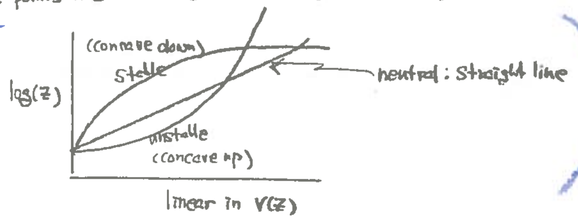
(C) Simple answer from Stull, p377

For neutral (adiabatic) conditions in the sfc layer we get the log profile for the wind

$$V(z) = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right)$$

When plotted on semi-log paper (linear in V, log in z), the logarithmic wind profile appears as a straight line. For non-neutral conditions, the wind profile deviates slightly from the logarithmic relationship above (see next answer of ⓒ). In stable sfc layers the wind profile is concave downward (indicating that sfc has less of an effect on the wind → shallower eddies). In unstable sfc layers the profile is concave upward (indicating the greater depth above the sfc which feels the effects of the sfc → deeper eddies).

These points are illustrated below (fig 9.5 in Stull)



(b)

$$1 = \frac{kz}{u_*}\frac{\partial V}{\partial z}$$

we recall that

$$\rho u_*^2 = \rho K_m \frac{\partial V}{\partial z} \quad \text{so} \quad K_m = u_*^2 \Big/ \frac{\partial V}{\partial z}$$

But from above

$$u_* = kz\frac{\partial V}{\partial z} \longrightarrow u_*^2 = (kz)^2\left(\frac{\partial V}{\partial z}\right)^2$$

Thus,

$$\boxed{K_m = (kz)^2\left(\frac{\partial V}{\partial z}\right)}$$

✓ (c) again ← from Pielke, section 7.3

When the atmosphere near the ground is not neutrally stratified, $S \neq 1$,

∴ To be consistent with Pielke we redefine the non-dimensional shear as

$$\phi_m = \frac{kz}{u_*}\frac{\partial V}{\partial z} \longrightarrow (1) \quad : \text{nondimensional wind shear.}$$

We must generalize the log wind profile $V(z) = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right)$ to include buoyancy effects. We start with the flux Richardson number,

$$R_f = \left(-\frac{g}{\theta_0}\overline{w'\theta'}\right)\Big/ u_*^2\left|\frac{\partial V}{\partial z}\right| \qquad (2)$$

Multiply (1) and (2)

$$R_f\phi_m = \left(-\frac{g}{\theta_0}\overline{w'\theta'}\Big/ u_*^2\frac{\partial V}{\partial z}\right)\left(\frac{kz}{u_*}\frac{\partial V}{\partial z}\right)$$

$$R_f\phi_m = -\frac{g}{\theta_0}\overline{w'\theta'}\cdot\frac{kz}{u_*^3}$$

$$R_f\phi_m = \left(-\frac{g}{\theta_0}\frac{\overline{w'\theta'}}{u_*^3}k\right)z$$

or

$$R_f\phi_m = \frac{z}{L} \qquad (3)$$

$$\text{where } L = \frac{-\theta_0 u_*^3}{gk\,\overline{w'\theta'}} \qquad (4)$$

units of L

$$\left(\frac{K\left(\frac{m}{s}\right)^3}{\frac{m}{s^2}\,\frac{m}{s}\,K} = m\right)$$

The parameter L has units of length (as it must since Re + ϕ_m are nondimensional) We call L the Monin or Monin-Obukhov length. We may view the non-dimensional wind shear ϕ_m as a function of z/L. That is, $\phi_m = \phi_m(\frac{z}{L})$. When $\frac{z}{L} < 0$ the atmosphere is unstably stratified and $\phi_m < 1$.

$\frac{z}{L} < 0$ implies $\overline{w'\theta'} > 0 \rightarrow$ warm air rising, cold air sinking.

For a stable stratification $\overline{w'\theta'} < 0 \rightarrow$ cold air rising, warm air sinking, $\frac{z}{L} > 0$ and $\phi_m > 1$.

For neutral stratification $z/L = 0$ and $\phi_m = 1$.

The modification to the log wind profile during non-neutral conditions may be derived from (1) by rewriting it as

$$\frac{kz}{u_*}\frac{\partial V}{\partial z} = 1 - (1 - \phi_m)$$

$$\text{or } \frac{\partial V}{\partial z} = \frac{u_*}{kz} - \frac{(1-\phi_m)}{kz}u_* \qquad (5)$$

We integrate (5) w.r.t. z from z_0 to a general level z

$$V(z) - \underbrace{V(z_0)}_{\substack{=0 \text{ by definition} \\ \text{of } z_0}} = \int_{z_0}^{z} \frac{u_*}{k}\frac{1}{z}dz - \int_{z_0}^{z} \frac{(1-\phi_m)}{kz'}u_*\,dz'$$

$\longrightarrow$ assume u_* is const w/ height.

$$V(z) = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) - \frac{u_*}{k}\int_{z_0}^{z}(1-\phi_m)\frac{dz'}{z'}$$

$\downarrow$

make a change of variables

Let $z' = \frac{\tilde{z}}{L} \Rightarrow dz' = \frac{d\tilde{z}}{L}$

at $z' = z \Rightarrow \tilde{z} = z'L = zL$ so upper limits becomes $\frac{z}{L}$

and lower limit becomes $\frac{z_0}{L}$ $\longrightarrow$ assume L is const. w/ height.

$$V(z) = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) - \frac{u_*}{k}\int_{\frac{z_0}{L}}^{\frac{z}{L}}(1-\phi_m)\,d\ln\left(\frac{\tilde{z}}{L}\right) \qquad (6)$$

This result is often written as

$$\underline{V(z) = \frac{u_*}{k}\left[\ln\left(\frac{z}{z_0}\right) - \Psi_m\left(\frac{z}{L}\right)\right]} \qquad (7)$$

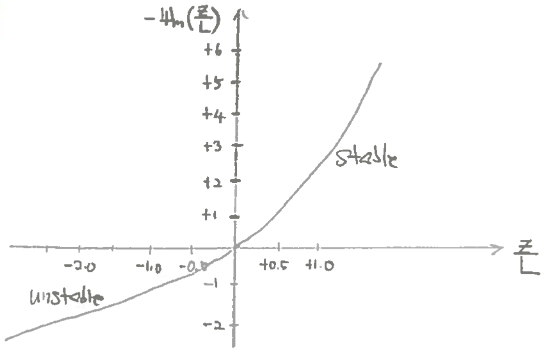
where we assume $\frac{z}{L} \gg \frac{z_0}{L} \sim 0$ and define

$$\Psi_m\left(\frac{z}{L}\right) = \int_0^{\frac{z}{L}} \frac{(1-\phi_m)}{\tilde{z}/L}\,d\left(\frac{\tilde{z}}{L}\right)$$

The parameter $\Psi_m(\frac{z}{L})$ is referred to as the correction to the logarithmic wind profile resulting from deviations from neutral stratification. For neutral stratification $\Psi_m = 0$ (obviously)

The following figure from Hartmann + Williams (Fig 8.4) provides a qualitative feel for $\Psi_m(\frac{z}{L})$. Note that the plotted function is $-\Psi_m(\frac{z}{L})$ so when the curve is below zero the wind is less than the logarithmic value (neutral). When $-\Psi_m(\frac{z}{L}) > 0$, the wind is greater than the logarithmic value (").

☆



From $\tau = \rho K_m \frac{\partial V}{\partial z}$; $\tau = \rho u_*^2$; and $\frac{\partial V}{\partial z} = \frac{u_*}{kz}\phi_m$

$$K_m \frac{\partial V}{\partial z} = u_*^2$$

$$K_m\left(\frac{u_*}{kz}\right)\phi_m = u_*^2$$

$$\boxed{K_m = \frac{u_* kz}{\phi_m}}$$

Since $K_m = \ell^2 \frac{\partial V}{\partial z}$,

$$\frac{u_* kz}{\phi_m} = \ell^2 \frac{u_*}{kz}\phi_m \longrightarrow \boxed{\ell = \frac{kz}{\phi_m}}$$

* Vort. eq. for a barotropic wind-driven ocean

OCP5551

MET? (Air-Sea Interaction): O'Brien (30 minutes, 1997) *

- The vorticity equation for a barotropic wind-driven ocean may be written as

$$\mathbf{V}\cdot\nabla\zeta+(\zeta+f)\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)+\beta V=\left(\mathbf{k}\cdot\nabla\times\frac{\tau}{\rho h}\right)+A_H\nabla^2\zeta-R\zeta$$

where $\mathbf{V}=U\mathbf{i}+V\mathbf{j}$,

U and V are the eastward and northward transports,

τ is the wind stress **vector**,

h is the depth of the fluid,

✓ R is the bottom friction coefficient, and

✓ A_H is the horizontal "Austausch" mixing coefficient.

a) What terms in the above constitute the Sverdrup balance?

b) What terms did Stommel use to explain westward intensification? Why did he need the additional term?

c) What terms did Monk use for his famous 1950 paper of the wind-driven ocean circulation?

d) There are some terms in the above which do not appear in any of these three simple models. When would these terms be important?

e) Suppose that the wind stress vector only has an x-component which varies in the y direction, i.e., $\tau=\overline{\tau_x}(y)\mathbf{i}$ with $\tau_y=0$. How would you calculate the Sverdrup transport for an ocean? (Use the continuity equation.)

based on Inf answer (He got 9/10)

Sol)

a) Sverdrup balance

✓ $$\boxed{\beta V=\mathbf{k}\cdot\nabla\times\frac{\tau}{\rho h}}$$

b) Stommel balance

$$\beta V=\mathbf{k}\cdot\nabla\times\frac{\tau}{\rho h}-R\zeta$$

The additional term used here is the ocean bottom friction term. This term, in order to be balanced, requires the current vorticity be different at the South and North of the west boundary and thus causing the westward intensification

c) Monk balance

$$\beta V=\mathbf{k}\cdot\nabla\times\frac{\tau}{\rho h}+A_H\nabla^2\zeta$$

d)
- advection term $\underline{V}\cdot\nabla\zeta$ will be important when there is current gyre is moved to the local along with the current, which can cause the local current vorticity change.
- developing (or divergent) term $(\zeta+f)\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)$ will be important when there is water divergence or convergence in which the area of water will enlarge or shrink, thus deduces the vorticity change

e) $\because\ \tau=\overline{\tau_x}(y)\mathbf{i}$ and $\tau_y=0$

$$\therefore\ \mathbf{k}\cdot\nabla\times\frac{\tau}{\rho h}=\frac{1}{\rho h}\mathbf{k}\cdot\nabla\times(\mathbf{i}\overline{\tau_x})=-\frac{\partial\overline{\tau_x}/\partial y}{\rho h}\qquad\because\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ \overline{\tau_x}&0&0\end{vmatrix}$$

$$\therefore\ \beta V=-\frac{1}{\rho h}\frac{\partial\overline{\tau_x}}{\partial y}$$

$$\Rightarrow V=-\frac{\partial\overline{\tau_x}/\partial y}{\beta\rho h}$$

$$\because\ \frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0\Longrightarrow U=\frac{\int\partial^2\overline{\tau_x}/\partial y^2\,dx}{\beta\rho h}\qquad\because\text{ assume }\beta=\text{const}$$

for unit length in x-direction, we can write

$$\begin{cases}U=\dfrac{1}{\beta\rho h}\dfrac{\partial^2\tau_x}{\partial y^2}\\ V=-\dfrac{1}{\beta\rho h}\dfrac{\partial\tau_x}{\partial y}\end{cases}$$

J.J. O'BRIEN

(1 hour)

?

Derive the barotropic vorticity equation for a wind-driven homogeneous ocean. The fluid is viscous, with horizontal Laplacian friction and vertical friction. Integrate from the surface down to depth h. Take h uniform and being either the bottom of the ocean or the base of the mixed layer. $\underset{\sim}{V}\cdot\nabla\zeta+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\beta V=\left(\mathbb{k}\cdot\nabla\times\frac{\underline{\tau}}{\rho h}\right)+A_H\nabla^2\zeta-R\zeta$

In the resulting equations, identify the balance of terms for

a) Sverdrup balance, $\beta V=\mathbb{k}\cdot\nabla\times\frac{\underline{\tau}}{\rho h}$

b) Stommel ocean-circulation model, $\beta V=\mathbb{k}\cdot\nabla\times\frac{\underline{\tau}}{\rho h}-R\zeta$

c) Munk ocean-circulation model, $\beta V=\mathbb{k}\cdot\nabla\times\frac{\underline{\tau}}{\rho h}+A_H\nabla^2\zeta$

✓ d) Ekman pumping.

* Sverdrup transport eqs

OCP5551

MET? (Air-Sea): O'Brien (30 minutes) *

- In this question, I would like you to discuss the Sverdrup transport equations. The equations you need are

$$\beta M_y = \mathbf{k}\cdot\mathrm{Curl}\underset{\sim}{\tau} = \mathbf{k}\cdot\nabla\times\underset{\sim}{\tau} = \frac{\partial\tau_y}{\partial x} - \frac{\partial\tau_x}{\partial y} \qquad (1)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0 \qquad (2)$$

where $\beta = \partial f/\partial y$, M_x and M_y are eastward and northward transports, respectively, and τ is the wind-stress vector. $\longrightarrow \underset{\sim}{\tau} = \tau_x \mathbf{i} + \tau_y \mathbf{j}$

(a) Suppose you were given $\underset{\sim}{\tau} = \tau_x(y)\mathbf{i}$, i.e., $\tau_y = 0$ for an ocean with lateral boundaries. How would you calculate the Sverdrup transports, M_x and M_y?

(b) The second equation above permits the introduction of a streamfunction for the mass-transport. Define the streamfunction.

V (c) Why do most oceanographers present calculations as you have outlined in terms of a streamfunction rather than as graphs of M_x and M_y?

(d) How would you use your calculations of M_x and M_y to compute the streamfunction? (Hint: Do not use the basic definition of the streamfunction.)

(a)

Since we are working in a bounded ocean we set the transports Mx and My to zero along these boundaries. We will use this fact later.

With $\tau_x = \tau_x(y)$ and $\tau_y = 0$ we obtain My directly from (1) as

$$\beta M_y = -\frac{\partial\tau_x}{\partial y}$$

We could finite difference $\frac{\partial\tau_x}{\partial y}$ if we are dealing with observations of τ_x rather than an analytic function. Whether we are working with continuous τ_x or discrete τ_x we can obtain My (provided we know $\beta = \frac{df}{dy}$ ⇒ we must know our position).

Given My we compute $\frac{\partial M_y}{\partial y}$. From (2) $\frac{\partial M_x}{\partial x} = -\frac{\partial M_y}{\partial y}$ so integration of this eq over x will give Mx. At the boundaries we use the imposed BCs of zero transport (or whatever BCs we impose)

(b)

we define ψ such that

$$M_x = -\frac{\partial\psi}{\partial y} \quad \text{and} \quad M_y = \frac{\partial\psi}{\partial x} \qquad (3)$$

Then

$$\frac{\partial M_x}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial\psi}{\partial y}$$

$$+)\quad \frac{\partial M_y}{\partial y} = +\frac{\partial}{\partial y}\frac{\partial\psi}{\partial x}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0$$

(c)

By defining a streamfunction ψ as we have done above, we can view the transport simply by looking at the ψ field rather than the Mx + My field together.

(d)

Having defined ψ above in (3), note that

$$\nabla^2\psi = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}$$

This is a Poisson eq.

Given fields of My and Mx we can evaluate the forcing. Given the forcing and BC on ψ we can invert the ∇^2 operator to obtain the ψ field.

→ If we are doing this numerically on a grid we might use some form of relaxa to solve for ψ.

Notes from Dr. O'Brien's NWP note

In a classical paper, Sverdrup (1947) showed that the time-independent wind stress is related to the northward mass transport in the ocean. Thus if we have an estimat the wind field, we can calculate the steady mass transport field. Since this vector is nondivergent, the scalar mass transport streamfunction is a convenient grap representation of the Sverdrup solution.

The Sverdrup model promblem is

$$\beta V = \mathbf{k}\cdot\nabla\times\frac{\underset{\sim}{\tau}}{\rho} = \frac{\partial}{\partial x}\left(\frac{\tau_y}{\rho}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{\rho}\right) \qquad (1)$$

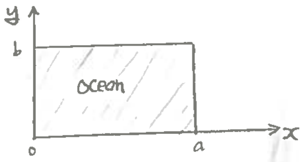
$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \qquad (2)$$

where U + V are vertically averaged mass transport (in my response Mx=U,

Eq (2) allows us to define a mass transport streamfunction ψ such that

$$\frac{\partial\psi}{\partial y} = -U \quad \text{and} \quad \frac{\partial\psi}{\partial x} = V \qquad (3)$$

For simplicity assume a rectangular basin $x \in [0, a]$, $y \in [0, b]$



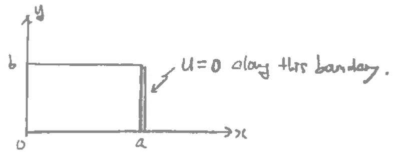
From a prescribed $\underset{\sim}{\tau}$ we can compute V everywhere using (1)

$$V = \beta^{-1}\left[\frac{\partial}{\partial x}\left(\frac{\tau_y}{\rho}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{\rho}\right)\right] \qquad (4)$$

Then U can be computed from (2) as

$$U(x,y) = \int_x^a \frac{\partial V}{\partial y}\,dx \qquad (5)$$

where we have used the BC $U(a, y) = 0$ ⇒ Along the 'eastern' boundary is no flow normal to the boundary.



From (3) we can construct a Poisson eq.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \nabla^2\psi$$

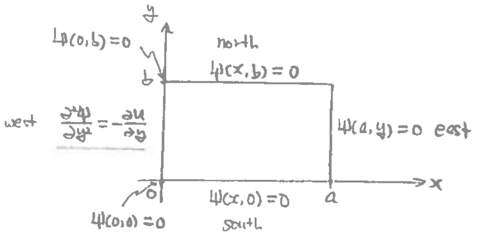
or $\nabla^2\psi = \zeta$. (6)

To solve (6) we need BCs on all boundaries. For this simple example we choose the northern & Southern boundary to be on a zero in $(\nabla\times\frac{\tau}{\rho})$. Thus $\psi=0$ is appropriate. On the western boundary we have to approximate the BC by solving

$$\frac{\partial^2\psi}{\partial y^2} = -\frac{\partial u}{\partial y} \qquad \text{where } \psi(0,0) = 0 = \psi(0,b)$$

This can be solved using a <u>tridiagonal algorithm</u>. Finally, on the eastern boundary, $U=0$ so $\psi=0$.

Our BC's on ψ are as shown below



We approx the domain with a uniform distance finite difference grid, $\Delta x = \Delta y = d$. We let

$$x_i = (i-1)d \quad \text{where } i=1,2,\cdots,I \text{ and } I-1 = \frac{a}{d}$$
$$y_j = (j-1)d \quad \text{"} \quad j=1,2,\cdots,J \text{ and } J-1 = \frac{b}{d}$$

At each grid point we approx

$$\psi(id, jd) = \psi_{ij}$$

Using standard, centered, second order finite differences we approx (6) with

$$\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial x^2} = \frac{\psi_{i+1,j}+\psi_{i-1,j}+\psi_{i,j+1}+\psi_{i,j-1}-4\psi_{i,j}}{d^2} + O(d^2)$$

So that

$$\underline{\psi_{i+1,j}+\psi_{i-1,j}+\psi_{i,j+1}+\psi_{i,j-1}-4\psi_{i,j} = d^2\zeta_{ij}}$$

We chose to solve this using <u>relaxation</u> (an iterative scheme).

We illustrate the technique at point ij

Let $\psi_{i,j}^{0}$ denotes our first guess at the true solution at (i,j). When we put $\psi_{i,j}^{0}$ into the finite difference eq. it is unlikely that the LHS will equal $d^2\zeta_{i,j}$. Rather we expect

$$\underline{\psi_{i+1,j}+\psi_{i-1,j}+\psi_{i,j+1}+\psi_{i,j-1}-4\psi_{ij}^{0} = d^2\zeta_{i,j}+\varepsilon_{i,j}^{0}}$$

where $\varepsilon_{i,j}^{0}$ is the error or residual.

We can temporarily reduce ε_{ij} to zero by adjusting ψ_{ij}^{0}. We define a new value for $\psi_{i,j}$, call it $\psi_{i,j}'$ as

$$\underline{\psi_{i,j}' = \psi_{i,j}^{0} + \frac{1}{4}\varepsilon_{i,j}^{0}}$$

If we replace $\psi_{i,j}^{0}$ in the eq above with $\psi_{i,j}' - \frac{1}{4}\varepsilon_{i,j}^{0}$ we see that $\varepsilon_{i,j}^{0}$ falls out of the equation. In general we have the iteration eq.

$$\underline{\psi_{i,j}^{n+1} = \psi_{i,j}^{n} + \frac{1}{4}\varepsilon_{i,j}^{n}}$$

We can implement this iterative relaxation in two ways.

① <u>Simultaneous relaxation</u> → correct $\varepsilon_{i,j}^{n}$ for all (i,j) ↺ iterate
compute $\psi_{i,j}^{n+1}$ for all (i,j)
↓
slow

② <u>Successive relaxation</u> → at each i,j compute ε_{ij}^{n} ↺ iterate
compute ψ_{ij}^{n+1} this i,j
move on to next i,j
↓
faster
↳ can be made faster sometimes by replacing 1/4 with $\alpha/4$ where α the relaxation parameter $\begin{cases} \alpha > 1 \rightarrow \text{over relaxation} \\ \alpha < 1 \rightarrow \text{under "} \end{cases}$

O'Brien (30 min)

We choose to solve the Munk problem numerically because the forcing function is complicated. We wish to determine the steady solution ~~by~~ for the time dependent problem.

$$\nabla^2 \psi_t = \beta \psi_x + K \nabla^4 \psi + T(x,y)$$

a) Outline a numerical scheme to solve this problem. Devise initial conditions. Carefully discuss the first time step and the cycling procedure.

?

MET? : O'Brien (1 hour)

- Equatorial Kelvin Waves

1. Write down the linear equatorial beta-plane, inviscid reduced gravity equations (3). Identify your symbols.
2. Explain how to find the equations for a free equatorial Kelvin Wave.
3. Find or write down the free Kelvin Wave Solution.
4. Describe as many of the physical attributes (in words) that you can identify (at least five). These are based on your answers for 3.
5. How does the Kelvin Wave play a role in a warm EL Nino event?

See other notes ?

PhD. Question (from O'Brien)

✓ 1 (a) Write down the linear, reduced-gravity equations on an equatorial beta plane.

$$U_t = +\beta yv - g'h_x$$

$$V_t\ \cancel{U_t} = -\beta yu - g'h_y$$

$$h_t = -H(U_x + V_y)$$

These are free wave equations, i.e., no forcing. The equatorial Kelvin Wave is a solution of these equations. Find the solution.

✓(b) List 5 properties of forced Kelvin waves.

(c) Suppose there are 2 typhoons close to the equator west of the dateline. These will generate a Kelvin wave. What sign (up or down) will the typhoon generate? Estimate the time the wave takes to reach 32°N off the coast of the Calif-Mexico border.

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To: Dr A. Clarke

From: Dr J. J. O'BRIEN

Question for Bin Li AND Ming Lin

Air-Sea Interactions (1½ hour

Coastal Upwelling

a) Explain in words when you would expect to observe coastal upwelling along a coastline.

b) Write down the reduced-gravity equations for the simplest model of coastal upwelling.

c) Perform a simple ~~scale analy~~sis to deduce the horizontal length scale for the width of upwelling and the radius of deformation (same number). Deduce a time scale for the vertical velocity, $\frac{\partial h}{\partial t}$.

Explain physically why there is strong upwelling at the equator in the Pacific and Atlantic Ocean. Continue your argument to explain physically why an equatorial undercurrent occurs in both oceans.

ORY 2 2

Ph.D. Exam
Spring 1991

c Brier

For: Yu
Spitz
Shriver

1. This question is on oceanic upwelling.

a) Explain in words when you would expect to observe coastal upwelling along a coastline.

1 hr. b) Given the reduced gravity equations

$$\frac{\partial u}{\partial t} = f\upsilon - c^2 \frac{\partial h}{\partial x}$$

$$\frac{\partial \upsilon}{\partial t} = fu - c^2 \frac{\partial h}{\partial y} + \tau^y/\rho H$$

$$\frac{1}{H}\frac{\partial h}{\partial t} = - u_x - \upsilon_y$$

with $c^2 = g'H$. Assume no variation in y and constant wind stress and no inertial oscillations. Calculate the solution for the vertical velocity, $w = \frac{\partial h}{\partial t}$ along a straight north-south coastline. From your solution estimate a typical value of w given reasonable parameters.

c) Equatorial Kelvin waves. Assume $f = \beta y$ and $\upsilon = 0$. Solve (b) for free Kelvin waves. List in words three important physical properties of equatorial Kelvin waves.

Ph.D. Preliminary 1995
Air-Sea Interaction

Air-Sea Interaction (O'Brien) - 1 Hour

The linear reduced gravity shallow water equations are:

$$u_t = fv - g' h_x$$

$$v_t = -fu - g' h_y$$

$$h_t = -H\left(u_x + v_y\right)$$

a) Consider an initial value problem along a straight N-S coast. The wind is from the north and begins at t=0 and is independent of space. If $f=10^{-4}$, $g'H=1$ and the wind stress, $U_*=0.1$m/sec, determine how long it takes h to change by 50m. Choose typical values for any density parameters you need.

b) Let the forcing be zero and find the characteristic solution for coastal kelvin waves (Hint: let u=0)

?

Dr O'Brien (Dynamics)

45 min 6. The usual form of the Navier-Stokes equations for a compressible fluid is

$$\rho \frac{du_i}{dt} = - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2}{\partial x_j \partial x_j} u_i + \frac{1}{3} \mu \frac{\partial}{\partial x_i} \textcircled{\#} + F_i$$

where u_i is the velocity; p, the pressure; μ, the dynamic viscosity; F_i, the forcing terms; and

$$\textcircled{\#} = \frac{\partial u_j}{\partial x_j}$$

(a) What is the fundamental hypothesis (or assumption) used to derive the Navier-Stokes equations? (Hint: the answer is not $\hat{F} = m\hat{A}$)

(b) What is the Reynolds stress tensor? How does it appear in the above equation after the equation is Reynolds averaged? Show that the Reynolds stress tensor is symmetric.

30 min ~~7. The log-linear profile equations for the atmospheric boundary layer are~~

$$v(z) = \frac{U^*}{k} \left[\ln (z/z_0) + \beta z/L \right] \quad (1)$$

$$T - T_0 = \frac{T^*}{k} \left[\ln (z/z_0) + \beta \frac{z}{L} \right] \quad (2)$$

~~where~~

$$L = \frac{U^{*3}}{\frac{g}{T} \frac{H}{\rho c_p}} \quad \text{(Monin length scale)}$$

~~(a) How would you determine U*, z_0, β, T_0, T* from actual meteorological data? A graphical technique is adequate. Assume that you can only measure v and T on a mast. Determine H, the heat flux from~~

$$H = - \rho c_p K_H \frac{\partial T}{\partial z}$$

~~where $K_H = K_M$. Note, k = 0.4 (von Karman's constant).~~

~~(b) Since $\tau = \rho U^{*2}$ and $\tau = K_M \frac{\partial v}{\partial z}$. Derive an expression for K_M which is consistent with (1). The answer should be a profile with height of K_M containing z and constants for the particular situation. Hint: The answer does not contain v.~~