MET5541: Krishnamurti

- Given the following time differencing for a wave equation, $u^{n+1} = u^n + \Delta t\left(\alpha f^n + \beta f^{n+1}\right)$ ——— (1)
where $f = -ikcu = i\omega u$, discuss the computational stability of this scheme for $\alpha = 0.25$ and $\beta = 0.75$ (Hint: pages 40, 41).

Sol)

$f^n = i\omega u^n$, $f^{n+1} = i\omega u^{n+1}$

(1) →

$u^{n+1} = u^n + \Delta t(\alpha i\omega u^n + \beta i\omega u^{n+1})$

$u^{n+1} = u^n \dfrac{1+i\alpha\omega\Delta t}{1-i\beta\omega\Delta t}$

Hence, the amplification factor λ,

$$\lambda = \frac{1+i\alpha\omega\Delta t}{1-i\beta\omega\Delta t}$$

Let $p=\omega\Delta t$ and multiplying by the conjugate of $(1-i\beta p)$

$$\lambda = \frac{1-\alpha\beta p^2+ip(\alpha+\beta)}{1+\beta^2p^2} \qquad \because \alpha+\beta=1$$

$$\therefore \lambda = \frac{1-\alpha\beta p^2+ip}{1+\beta^2p^2}$$

By assigning $\alpha = 0.25 = \frac{1}{4}$, $\beta = 0.75 = \frac{3}{4}$

$$\lambda = \frac{1-\frac{3}{16}p^2+ip}{1+\frac{9}{16}p^2} = \frac{16-3p^2+16ip}{16+9p^2}$$

$$\therefore |\lambda| = \frac{\sqrt{256+9p^4-96p^2+256p^2}}{16+9p^2}$$

$$= \frac{\sqrt{256+160p^2+9p^4}}{16+9p^2}$$

here $\sqrt{256+160p^2+9p^4} < 16+9p^2 = \sqrt{256+288p^2+81p^4}$ for any p.

Therefore $|\lambda| < 1$ → always stable.

MET5541: Krishnamurti (1998)

- If U, V and ϕ' are given on a grid space (λ,θ) and you are to compute the spectral coefficient of $-\frac{1}{\cos^2\theta}\left(\frac{\partial U\phi'}{\partial\lambda}+\cos\theta\frac{\partial V\phi'}{\partial\theta}\right)$ (of equation 6.132) what steps would you take. Write down all steps and provide very brief explanation of each step. Note: U, V are Robert functions of u, v.

Sol. nonlinear term

$-\frac{1}{\cos^2\theta}\left(\frac{\partial U\phi'}{\partial\lambda}+\cos\theta\frac{\partial V\phi'}{\partial\theta}\right) \longrightarrow$ compute the spectral coefficient.

$$= -\underbrace{\frac{\partial}{\partial\lambda}\left(\frac{U\phi'}{\cos^2\theta}\right)}_{\mathrm{I}} - \underbrace{\frac{1}{\cos\theta}\frac{\partial V\phi'}{\partial\theta}}_{\mathrm{II}} \equiv -\alpha(U\phi', V\phi')$$

where $\alpha(A,B) = \frac{1}{\cos^2\theta}\left(\frac{\partial A}{\partial\lambda}+\cos\theta\frac{\partial B}{\partial\theta}\right)$

$U = \frac{u\cos\theta}{a}$, $V = \frac{v\cos\theta}{a}$

Term I

$U(\lambda,\theta)$ $\phi'(\lambda,\theta)$

$\frac{U\phi'}{\cos^2\theta}$: multiply on the grid space (λ,θ)

$\downarrow$ do Fourier-Legendre transform

$\left\{\frac{U\phi'}{\cos^2\theta}\right\}_n^m$

$\downarrow$

$-\frac{\partial}{\partial\lambda}\left\{\frac{U\phi'}{\cos^2\theta}\right\}_n^m \longrightarrow$ compute spectrally.

Term II

$V(\lambda,\theta)$, $\phi'(\lambda,\theta)$

$V\phi'$: multiply on the grid space (λ,θ)

$\downarrow$ do Fourier-Legendre transform

$\{V\phi'\}_n^m$

$\downarrow$

$-\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\{V\phi'\}_n^m = -\frac{\partial\{V\phi'\}_n^m}{\partial\mu}$ where $\mu=\sin\theta$

$\hookrightarrow$ compute spectrally using the recurrence relations.

$\therefore$ The spectral coefficient of $\alpha(U\phi', V\phi')$, i.e $\alpha_n^m(U\phi', V\phi')$ can be obtained

MET5541: Krishnamurti (1998)

- Explain the method (on pages 160, 161) for computing the nonconvective heating and nonconvective rain.

a) State all assumptions

b) Why are we making such a fuss about $\frac{\partial q_s}{\partial p}$

c) Provide a flow chart.

※ must look at p190

Sol) Large-scale condensation is usually invoked if dynamic ascent of absolute stable saturated air occurs

The conditions for large-scale condensation can be described as follows,

$$\begin{cases} \omega < 0 & \longrightarrow \text{ascent} \\ -\frac{\partial \theta}{\partial p},\ -\frac{\partial \theta_e}{\partial p} > 0 & \longrightarrow \text{stable} \\ q/q_s > 0.8 \text{ or } 1.0 & \longrightarrow \text{saturation} \end{cases}$$

Under the above conditions, we can compute the nonconvective heating and rain.

- method 1 : disposition of supersaturation (simple scheme)

$$C_p \frac{T}{\theta}\frac{\partial \theta}{\partial t} = + \frac{L \Delta q}{\Delta t} \quad : \text{the first law of thermodynamics}$$

$$\frac{\partial q}{\partial t} = -\frac{\Delta q}{\Delta t} \quad : \text{the moisture continuity eq.}$$

where $\Delta q = q - q_s$

→ The supersaturation is simply condensed out with an equivalent heat release

→ Calculation of the q_s is usually carried out by Teten's formula

$$\begin{cases} e_s = 6.11 \exp\left(\frac{a(T-273.16)}{T-b}\right) & \text{——— (a)} \\ q_s = \frac{0.622 e_s}{p} & \text{——— (b)} \end{cases}$$

→ The simplest formulation for nonconvective heating is

$$H_{NC} = -L\frac{dq_s}{dt} \approx -L\omega \frac{\partial q_s}{\partial p} \quad \text{——— (※)}$$

where $\frac{\partial q_s}{\partial p}$ is calculated along a moist adiabat

└ difficult to compute!

→ In order to calculate $\frac{\partial q_s}{\partial p}$, we utilize the conservation of moist static energy along a moist adiabat and Teten's formula.

moist static energy

$$E_s = gZ_s + C_p T_s + L q_s$$

↓ differentiating with respect to pressure

$$\frac{\partial E_s}{\partial p} = g\frac{\partial Z_s}{\partial p} + C_p \frac{\partial T_s}{\partial p} + L\frac{\partial q_s}{\partial p} = 0 \quad \text{along a moist adiabat}$$

or

$$-\frac{RT_s}{p}(1+0.61 q_s) + C_p \frac{\partial T_s}{\partial p} + L\frac{\partial q_s}{\partial p} = 0 \quad \text{——— (1)}$$

↳ one equation has two unknowns $\left(\frac{\partial T_s}{\partial p}, \frac{\partial q_s}{\partial p}\right)$

So, from Teten's formula, given T and p, Tetens gives us e_s and q_s.

$$\frac{\partial}{\partial p}(b) \rightarrow \frac{\partial q_s}{\partial p} = \frac{0.622}{p-0.378 e_s}\frac{\partial e_s}{\partial p} - \frac{0.622 e_s}{(p-0.378 e_s)^2}\left(1 - 0.378\frac{\partial e_s}{\partial p}\right)$$

$$\frac{\partial}{\partial p}\ln(a) \rightarrow \frac{1}{e_s}\frac{\partial e_s}{\partial p} = \frac{a\frac{\partial T}{\partial p}}{T-b} - \frac{a(T-273.16)}{(T-b)^2}\frac{\partial T}{\partial p}$$

So, we have the following form of equation

$$A\frac{\partial T}{\partial p} + B\frac{\partial q_s}{\partial p} = C \quad \text{——— (2)}$$

From (1) and (2), eliminating $\frac{\partial T_s}{\partial p}$, we can obtain $\frac{\partial q_s}{\partial p}$

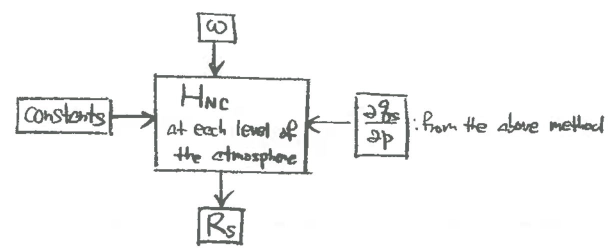
→ Therefore, the H_{NC} (eq. ※) can be computed.

→ nonconvective rain

$$R_s = \frac{1}{\rho_w g}\int_0^{p_s} \frac{H_{NC}}{L}\,dp \qquad \text{where } \rho_w : \text{water density}$$

So, if we knows H_{NC}, we can easily compute R_s

→



- method 2 A scheme based on the relative humidity equation

→ The heating is obtained by seeking a condition based on the local change of relative humidity set to zero.

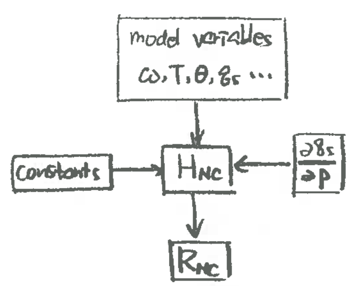
(Full eqs for moisture and thermodynamics)

→ nonconvective heating

$$H_{NC} = \frac{1}{C_p\frac{T}{\theta L q_s} + \frac{\epsilon L}{RT\theta}}\left[-\frac{\omega}{q_s}\frac{\partial q_s}{\partial p} + \cdots\right]$$

→ nonconvective rain

$$R_{NC} = \frac{1}{\rho_w g}\int_{p_T}^{p_s} \frac{H_{NC}}{L} C_p \frac{T}{\theta}\,dp \quad : \text{total convective rain}$$



MET5541: Krishnamurti (1998)

- In the surface similarity theory for the unstable constant flux layer we came across the following three equations (pages 140, 141)

$$\frac{kz}{u^*}\frac{\partial \bar{u}}{\partial z} = \left(1 - 15\frac{z}{L}\right)^{-1/4} \qquad (1)$$

$$\frac{kz}{\theta^*}\frac{\partial \bar{\theta}}{\partial z} = 0.74\left(1 - 9\frac{z}{L}\right)^{-1/2} \qquad (2)$$

$$L = u^{*2} / k\beta\theta^* \qquad (3)$$

These are three equations for the three unknowns L, θ^* and u^*.

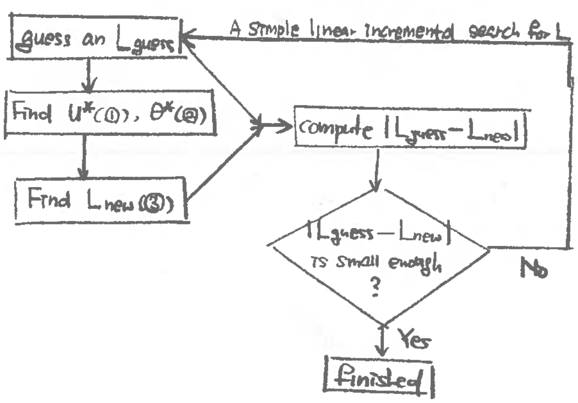
a) Describe a simple solution procedure (in words) for the above 3 equations.

b) Once you have solved for u^*, θ^* and L, how do you find the flux of momentum and heat (hint: page 141).

a) a simple solution procedure

The variation of the Monin-Obukhov length is monotonic with respect to u^* and θ^* (Businger et al, 1991). A simple linear incremental search of L in equation (3) provides a rapid solution to the desired degree of accuracy. After substitution for L on the RHS of eqs (1) and (2), one can obtain the corresponding solution for u^* and θ^*.

Guess an L_{guess} and substitute it into (1) and (2), find u^* and θ^*. (where $\frac{\partial \bar{u}}{\partial z}$, $\frac{\partial \bar{\theta}}{\partial z}$ are knowns) and using these u^* and θ^* in eq. (3), find L_{new}, check the difference between L_{guess} and L_{new}. If the difference is small enough, we find the answers. If not, with another L_{guess}, repeat the above procedures.



b) The surface fluxes of momentum and heat are, respectively, defined by

$$F_M = u^{*2} \equiv -\overline{u'w'}|_0$$

$$F_H = -u^*\theta^* \equiv \overline{\theta' w'}|_0$$

Therefore, just substituting the u^* and θ^*, we can easily find the fluxes of momentum and heat.

MET5541: Krishnamurti

• Question

a) Explain with diagrams the triangular truncation for the expansion of a variable $A(\mu,\lambda)$ using spherical harmonics.

b) Starting from the wind components u, v on grid array at 500 mb over the globe, show how you would obtain spectrally the following fields over the globe:
   a) Vorticity,
   b) Divergence,
   c) Streamfunction,
   and d) Velocity potential

$$l(l+1)U_l^m = -(l+1)\varepsilon_l^m \zeta_{l-1}^m + l\varepsilon_{l+1}^m \zeta_{l+1}^m - imD_l^m$$

$$l(l+1)V_l^m = +(l+1)\varepsilon_l^m D_{l-1}^m - l\varepsilon_{l+1}^m D_{l+1}^m - im\zeta_l^m$$

$$\varepsilon_l^m = \sqrt{\left(l^2 - m^2\right)/\left(4l^2 - 1\right)}$$

MET5541: Krishnamurti

• Given the following equations on radiative fluxes (from your class notes) interpret the meanings of these equations for a reference level above a single cloud layer.

Longwave

$$F_i\uparrow = \sigma T_{CT}^4\left(1-\varepsilon[W_{CT}-W_i]\right) - \int_{W_i}^{W_{CT}} \sigma T^4 \frac{\partial\varepsilon}{\partial W}[W-W_i]dW$$

net absorbed downward flux of SW radiation passing through a reference level i

Shortwave

the amount of short-wave radiation reaching a reference level i just above the cloud level

The upward diffuse radiation reaching the level i

$$S_i^a = \boxed{S^a\left(1-A[W_i\sec\zeta]\right)} - \boxed{S^a\left(1-A[W_{CT}\sec\zeta]\alpha_c\left(1-A[1.66(W_{CT}-W_i)]\right)\right)}$$

Show how one determines the cooling/warming rates for this situation over a layer between i and i+1 (both located above the cloud layer).

The diffuse radiation emanating upwards at the cloud top

Sol)

If there is one cloud layer below the reference level, i, then the formula for the upward flux of longwave radiative flux would be

$$F_i\uparrow = \sigma T_{CT}^4(1-\varepsilon(W_{CT}-W_i)) - \int_{W_i}^{W_{CT}} \sigma T^4 \frac{\partial\varepsilon(W-W_i)}{\partial W}dW$$

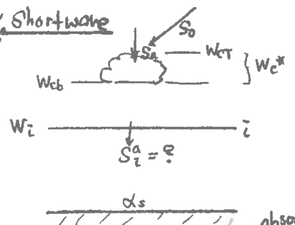
where
- $F_i\uparrow$: total upward flux of longwave radiation at a reference level
- σ : Stefan-Boltzmann constant
- T_{CT} : temperature at the cloud top
- ε : emissivity (function of the path length)
- W_i : the path length at the reference level
- W_{CT} : " " at the cloud top.

→ net long-wave radiative flux

$$F_i = F_i\downarrow - F_i\uparrow$$

$$\therefore C_p\frac{\partial T}{\partial t} = g\frac{\partial F}{\partial p}$$

※ Shortwave



α = albedo = $\frac{\text{reflected}}{\text{incident}}$

absorptivity function

A is a function of ...

augmented path length from liquid water and ice in the cloud

$$S_i^a = \boxed{S^a(1-A[W_{CT}\sec\zeta])} \times (1-\alpha_c) \times \{ (1-A[W_c^* + 1.66(W_i - W_{cb})]) - (1-A[W_c^* + 1.66(W_0 - W_{cb})]\alpha_s \times (1-A[1.66(W_0-W_i)]) \}$$

↑ albedo of cloud

Amount of absorved solar radiation that reaches cloud top

prorated by the cloud and up to reference level i

MET5541: Krishnamurti

• What is the purpose of normal mode initialization of a primitive equation NWP model? Given the linear form of equation of motion in sigma-coordinate:

$$\frac{\partial u'}{\partial t} - 2\Omega v' \sin\theta = -\frac{\partial P'}{a\cos\theta\partial\lambda}$$

$$\frac{\partial v'}{\partial t} + 2\Omega u' \sin\theta = -\frac{\partial P'}{a\partial\theta}$$

$$\frac{\partial W'}{\partial \sigma} + \nabla \cdot \mathbf{V}' = 0$$

$$\frac{\sigma}{R\Gamma_0}\frac{\partial}{\partial t}\left(\frac{\partial P'}{\partial \sigma}\right) + W' = 0$$

where u, v, P, W are functions of λ, θ, σ and t, primes indicate perturbation quantities and $P = gz + RT_0 \ln p_s$

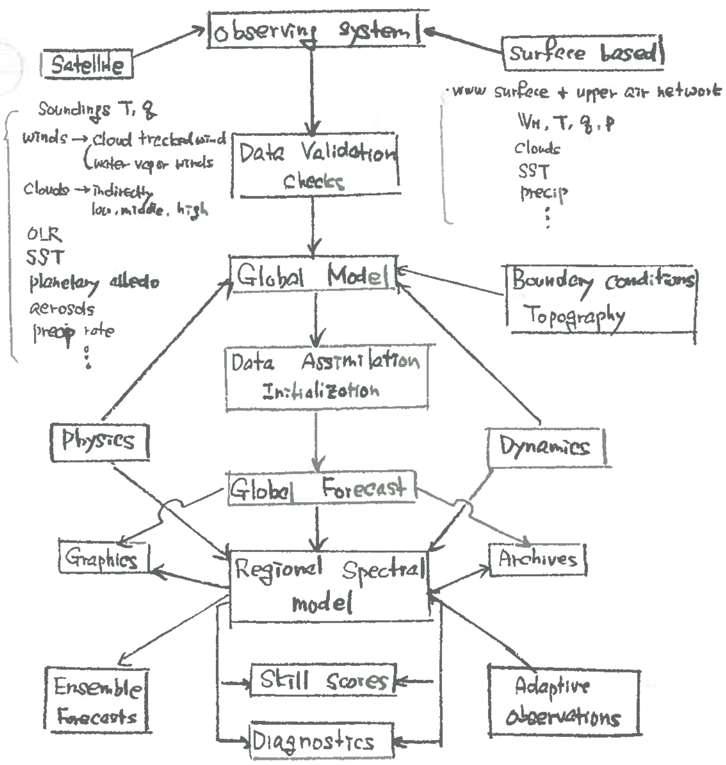
$$W = \dot{\sigma} - \sigma\left(\nabla \cdot \overline{\mathbf{V}} + \overline{\mathbf{V}} \cdot \nabla q\right) \qquad ; q = \ln p_s$$

Γ_0 = static stability of basic atmosphere

a) Derive the equations for vertical and horizontal structure of linear normal modes.
b) Describe briefly the method for solving vertical structure equation to obtain vertical modes.

< NWP Krish >

- Flowchart of NWP



- Taylor's expansion

$$u(x+\Delta x) = u(x) + \frac{du}{dx}\Big|_x \Delta x + \frac{d^2u}{dx^2}\Big|_x \frac{(\Delta x)^2}{2!} + \text{hot} \quad \text{——— ①}$$

$$u(x-\Delta x) = u(x) - \frac{du}{dx}\Big|_x \Delta x + \frac{d^2u}{dx^2}\Big|_x \frac{(\Delta x)^2}{2!} + \text{hot} \quad \text{——— ②}$$

$$u(x+2\Delta x) = u(x) + \frac{du}{dx}\Big|_x 2\Delta x + \frac{d^2u}{dx^2}\Big|_x \frac{(2\Delta x)^2}{2!} + \text{hot} \quad \text{——— ③}$$

$$u(x-2\Delta x) = u(x) - \frac{du}{dx}\Big|_x 2\Delta x + \frac{d^2u}{dx^2}\Big|_x \frac{(2\Delta x)^2}{2!} + \underbrace{\text{hot}}_{\text{truncation error}} \quad \text{——— ④}$$

- Forward + backward differencing (first order)

① →

$$\frac{du}{dx}\Big|_x \Delta x = u(x+\Delta x) - u(x) - \left(\frac{d^2u}{dx^2}\Big|_x \frac{(\Delta x)^2}{2!} + \text{hot}\right)$$

$$\frac{du}{dx}\Big|_x \cong \frac{u(x+\Delta x) - u(x)}{\Delta x} + O(\Delta x) \quad \text{: forward finite diff}$$

↖ truncation error

② →

$$\frac{du}{dx}\Big|_x \cong \frac{u(x) - u(x-\Delta x)}{\Delta x} + O(\Delta x) \quad \text{: backward finite diff}$$

- Centered finite differencing → $O(\Delta x)^2$

① − ② →

$$\frac{du}{dx}\Big|_x \cong \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} - O(\Delta x)^2$$

$$\frac{d^2u}{dx^2}\Big|_x = ?$$

① + ② →

$$u(x+\Delta x) + u(x-\Delta x) = 2u(x) + 2\frac{d^2u}{dx^2}\Big|_x \frac{(\Delta x)^2}{2!} + \text{hot}.$$

$$\frac{d^2u}{dx^2}\Big|_x \cong \frac{u(x+\Delta x) + u(x-\Delta x) - 2u(x)}{(\Delta x)^2} - O(\Delta x)^2$$

- Fourth-order accurate formulas (combination of ① to ④)

– First derivative

$$Au(x) + B[u(x+\Delta x) - u(x-\Delta x)] + C[u(x+2\Delta x) - u(x-2\Delta x)]$$
$$= \frac{du}{dx}\Big|_x \Delta x + O(\Delta x)^5 \quad \text{——— Ⓐ}$$

To determine A, B + C

① − ②

$$u(x+\Delta x) - u(x-\Delta x) = 2\frac{du}{dx}\Big|_x \Delta x + \frac{d^3u}{dx^3}\Big|_x \frac{(\Delta x)^3}{3} + \frac{d^5u}{dx^5}\Big|_x \frac{(\Delta x)^5}{60} + \cdots$$

③ − ④

$$u(x+2\Delta x) - u(x-2\Delta x) = 2\frac{du}{dx}\Big|_x 2\Delta x + \frac{d^3u}{dx^3}\Big|_x \frac{(2\Delta x)^3}{3} + \frac{d^5u}{dx^5}\Big|_x \frac{(2\Delta x)^5}{60} + \cdots$$

Ⓐ →

$$Au(x) + (2B+4C)\frac{du}{dx}\Big|_x \Delta x + (B+8C)\frac{d^3u}{dx^3}\Big|_x \frac{(\Delta x)^3}{3} + O(\Delta x)^5 \cong \frac{du}{dx}\Big|_x \Delta x$$

$\therefore A = 0, \quad 2B + 4C = 1, \quad B + 8C = 0$

$\rightarrow A = 0, \quad B = 2/3, \quad C = -1/12$

Ⓐ →

$$\frac{du}{dx}\Big|_x = 0u(x) + \frac{2}{3}\frac{u(x+\Delta x) - u(x-\Delta x)}{\Delta x} - \frac{1}{12}\frac{u(x+2\Delta x) - u(x-2\Delta x)}{\Delta x} + O(\Delta x$$

$$\therefore \frac{du}{dx}\Big|_x \cong \frac{4}{3}\frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} - \frac{1}{3}\frac{u(x+2\Delta x) - u(x-2\Delta x)}{4\Delta x}$$

– Second derivative

$$Au(x) + B[u(x+\Delta x) + u(x-\Delta x)] + C[u(x+2\Delta x) + u(x-2\Delta x)]$$
$$= \frac{d^2u}{dx^2}\Big|_x (\Delta x)^2 \quad \text{——— Ⓑ}$$

① + ② →

$$u(x+\Delta x) + u(x-\Delta x) = 2u(x) + \frac{d^2u}{dx^2}\Big|_x (\Delta x)^2 + \frac{d^4u}{dx^4}\Big|_x \frac{(\Delta x)^4}{12} + O(\Delta x)^6$$

③ + ④ →

$$u(x+2\Delta x) + u(x-2\Delta x) = 2u(x) + 4\frac{d^2u}{dx^2}(\Delta x)^2 + \frac{4}{3}\frac{d^4u}{dx^4}\Big|_x (\Delta x)^4 + O(\Delta x)^6$$

Ⓑ →

$$Au(x) + B\left(2u(x) + \frac{d^2u}{dx^2}\Big|_x (\Delta x)^2 + \frac{d^4u}{dx^4}\Big|_x \frac{(\Delta x^4)}{12} + O(\Delta x)^6\right)$$
$$+ C\left(2u(x) + 4\frac{d^2u}{dx^2}\Big|_x (\Delta x)^2 + \frac{4}{3}\frac{d^4u}{dx^4}\Big|_x (\Delta x)^4 + O(\Delta x)^6\right) \cong \frac{d^2u}{dx^2}\Big|_x (\Delta x)^2$$

$$(A + 2B + 2C)u(x) + (B + 4C)\frac{d^2u}{dx^2}\Big|_x (\Delta x)^2 + \left(\frac{B}{12} + \frac{4C}{3}\right)\frac{d^4u}{dx^4}\Big|_x (\Delta x)^4 + O(\Delta x)^6$$
$$\cong \frac{d^2u}{dx^2}\Big|_x (\Delta x)^2$$

$\therefore A + 2B + 2C = 0, \quad B + 4C = 1, \quad B/12 + 4C/3 = 0$

$\rightarrow A = -\frac{5}{2}, \quad B = \frac{4}{3} \quad C = -\frac{1}{12}$

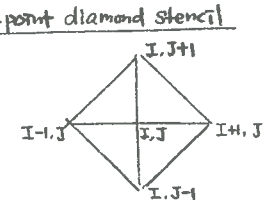
Ⓐ →

$$\frac{d^2u}{dx^2}\Big|_x = \frac{1}{(\Delta x)^2}\left(-\frac{5}{2}u(x) + \frac{4}{3}[u(x+\Delta x) + u(x-\Delta x)]\right.$$
$$\left. - \frac{1}{12}[u(x+2\Delta x) + u(x-2\Delta x)]\right) + O(\Delta x)^4$$

- Second-order Accurate Laplacian

Laplace's equation : $\nabla^2\psi = 0$, $\quad \psi = \psi(x, y)$

– A 5-point diamond stencil



Taylor's expansion

$$\psi(I+1, J) = \psi(I, J) + \frac{\partial\psi}{\partial x}\Big|_{I,J} \Delta x + \frac{\partial^2\psi}{\partial x^2}\Big|_{I,J} \frac{(\Delta x)^2}{2!} + \text{hot} \quad \text{——— ⓐ}$$

$$\psi(I-1, J) = \psi(I, J) - \frac{\partial\psi}{\partial x}\Big|_{I,J} \Delta x + \frac{\partial^2\psi}{\partial x^2}\Big|_{I,J} \frac{(\Delta x)^2}{2!} + \text{hot} \quad \text{——— ⓑ}$$

$$\psi(I, J+1) = \psi(I, J) + \frac{\partial\psi}{\partial x}\Big|_{I,J} \Delta y + \frac{\partial^2\psi}{\partial y^2}\Big|_{I,J} \frac{(\Delta y)^2}{2!} + \text{hot} \quad \text{——— ⓒ}$$

$$\psi(I, J-1) = \psi(I, J) - \frac{\partial\psi}{\partial x}\Big|_{I,J} \Delta y + \frac{\partial^2\psi}{\partial y^2}\Big|_{I,J} \frac{(\Delta y)^2}{2!} + \text{hot} \quad \text{——— ⓓ}$$

Let $\Delta x = \Delta y = \Delta$

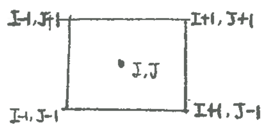
ⓐ+ⓑ+ⓒ+ⓓ →

$$\psi(I+1,J)+\psi(I-1,J)+\psi(I,J+1)+\psi(I,J-1)-4\psi(I,J) = \nabla^2\psi|_{I,J}\Delta^2 + 2\left(\frac{\partial^4\psi}{\partial x^4}\frac{\Delta^4}{4!}+\frac{\partial^4\psi}{\partial y^4}\frac{\Delta^4}{4!}\right)\Big|_{I,J} + \text{hot} \quad \text{— ⓘ}$$

$$\therefore \nabla^2\psi|_{I,J} \cong \frac{\psi(I+1,J)+\psi(I-1,J)+\psi(I,J+1)+\psi(I,J-1)-4\psi(I,J)}{\Delta^2}$$

↳ more accurate!

- A 5-point square stencil



$$\psi(I+1,J+1) = \psi(I,J) + \Delta\left(\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\right)\Big|_{I,J} + \frac{\Delta^2}{2!}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\Big|_{I,J} + \Delta^2\frac{\partial^2\psi}{\partial x\partial y}\Big|_{I,J}+\cdots$$

$$\psi(I-1,J-1) = \psi(I,J) - \Delta\left(\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\right)\Big|_{I,J} + \frac{\Delta^2}{2!}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\Big|_{I,J} + \Delta^2\frac{\partial^2\psi}{\partial x\partial y}\Big|_{I,J}+\cdots$$

$$\psi(I-1,J+1) = \psi(I,J) - \Delta\left(\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial y}\right)\Big|_{I,J} + \frac{\Delta^2}{2!}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\Big|_{I,J} - \Delta^2\frac{\partial^2\psi}{\partial x\partial y}\Big|_{I,J}+\cdots$$

$$+)\ \psi(I+1,J-1) = \psi(I,J) + \Delta\left(\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial y}\right)\Big|_{I,J} + \frac{\Delta^2}{2!}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\Big|_{I,J} - \Delta^2\frac{\partial^2\psi}{\partial x\partial y}\Big|_{I,J}+\cdots$$

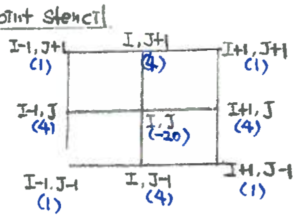
$$\psi(I+1,J+1)+\psi(I-1,J-1)+\psi(I-1,J+1)+\psi(I+1,J-1)-4\psi(I,J)$$
$$= 2\Delta^2\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\Big|_{I,J} + \frac{\Delta^4}{6}\left(\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^4\psi}{\partial y^4}+6\frac{\partial^4\psi}{\partial x^2\partial y^2}\right)\Big|_{I,J} + \text{hot}$$

By taking differences along the diagonal

$$\psi(I+1,J+1)+\psi(I-1,J-1)+\psi(I-1,J+1)+\psi(I+1,J-1)-4\psi(I,J)$$
$$= \nabla^2\psi(\sqrt{2}\Delta)^2 + 2\frac{(\sqrt{2}\Delta)^4}{4!}\nabla^4\psi+\cdots$$
$$= 2\nabla^2\psi|_{I,J}\Delta^2 + \frac{1}{3}\nabla^4\psi\Delta^4+\cdots \quad \text{— ⓘⓘ}$$

$$\therefore \nabla^2\psi = \frac{\psi(I+1,J+1)+\psi(I-1,J-1)+\psi(I-1,J+1)+\psi(I+1,J-1)-4\psi(I,J)}{2\Delta^2}$$

- A 9-point stencil



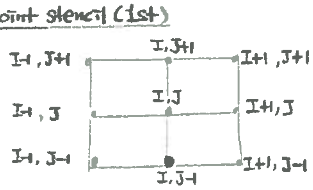
4*ⓘ + ⓘⓘ →

$$\psi(I+1,J+1)+\psi(I-1,J-1)+\psi(I-1,J+1)+\psi(I+1,J-1) + 4[\psi(I,J+1)+\psi(I-1,J)+\psi(I,J-1)+\psi(I+1,J)] - 20\psi(I,J)$$
$$= 6\Delta^2\nabla^2\psi|_{I,J} + \frac{2}{3}\Delta^4\nabla^4\psi|_{I,J} + \text{hot}$$

$$\therefore \nabla^2\psi|_{I,J} = \frac{1}{6\Delta^2}\{\psi(I+1,J+1)+\psi(I-1,J-1)+\psi(I-1,J+1)+\psi(I+1,J-1) + 4[\psi(I,J+1)+\psi(I-1,J)+\psi(I,J-1)+\psi(I+1,J)] - 20\psi(I,J)\} + O(\Delta^2)$$

• Fourth-order accurate Laplacian

- 9-point stencil (1st)



ⓘ →

$$\psi(I+1,J)+\psi(I-1,J)+\psi(I,J+1)+\psi(I,J-1)-4\psi(I,J)$$
$$= \nabla^2\psi|_{I,J}\Delta^2 + 2\frac{\Delta^4}{4!}\left(\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^4\psi}{\partial y^4}\right)\Big|_{I,J} + 2\frac{\Delta^6}{6!}\left(\frac{\partial^6\psi}{\partial x^6}+\frac{\partial^6\psi}{\partial y^6}\right)\Big|_{I,J} + \text{hot}$$

ⓘⓘ →

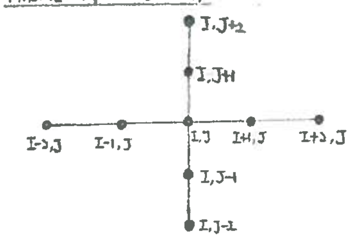
$$\psi(I-1,J+1)+\psi(I-1,J-1)+\psi(I+1,J-1)+\psi(I+1,J+1)-4\psi(I,J)$$
$$= \nabla^2\psi|_{I,J}(\sqrt{2}\Delta)^2 + 2\frac{(\sqrt{2}\Delta)^4}{4!}\left(\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^4\psi}{\partial y^4}\right)\Big|_{I,J} + 2\frac{(\sqrt{2}\Delta)^6}{6!}\left(\frac{\partial^6 x}{\partial x^6}+\frac{\partial^6 x}{\partial y^6}\right)\Big|_{I,J} + \text{hot}$$

4*ⓘ − ⓘⓘ

$$\nabla^2\psi|_{I,J} = \frac{1}{2\Delta^2}\{4[\psi(I,J+1)+\psi(I-1,J)+\psi(I,J-1)+\psi(I+1,J)]$$
$$- [\psi(I+1,J+1)+\psi(I-1,J+1)+\psi(I-1,J-1)+\psi(I+1,J-1)]$$
$$- 12\psi(I,J)\}$$
$$+ \frac{\Delta^4}{180}\left(\frac{\partial^6\psi}{\partial x^6}+\frac{\partial^6\psi}{\partial y^6}\right)\Big|_{I,J} + \text{hot}$$

$$\therefore \boxed{\nabla^2\psi|_{I,J} = \frac{1}{2\Delta^2}\{4[\psi(I,J+1)+\psi(I-1,J)+\psi(I,J-1)+\psi(I+1,J)] - [\psi(I+1,J+1)+\psi(I-1,J+1)+\psi(I-1,J-1)+\psi(I+1,J-1)] - 12\psi(I,J)\} + O(\Delta^4)}$$

- Another 9 point stencil



ⓘ →

$$\psi(I-1,J)+\psi(I,J-1)+\psi(I+1,J)+\psi(I,J+1)-4\psi(I,J)$$
$$= \nabla^2\psi|_{I,J}\Delta^2 + 2\frac{\Delta^4}{4!}\left(\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^4\psi}{\partial y^4}\right)\Big|_{I,J} + 2\frac{\Delta^6}{6!}\left(\frac{\partial^6\psi}{\partial x^6}+\frac{\partial^6\psi}{\partial y^6}\right)\Big|_{I,J} + \text{hot}$$

similarly

$$\psi(I,J+2)+\psi(I-2,J)+\psi(I,J-2)+\psi(I+2,J)-4\psi(I,J)$$
$$= \nabla^2\psi|_{I,J}(2\Delta)^2 + 2\frac{(2\Delta)^4}{4!}\left(\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^4\psi}{\partial y^4}\right)\Big|_{I,J} + 2\frac{(2\Delta)^6}{6!}\left(\frac{\partial^6\psi}{\partial x^6}+\frac{\partial^6\psi}{\partial y^6}\right)\Big|_{I,J} + \text{hot}$$

16*ⓐ − ⓑ

$$\therefore \boxed{\nabla^2\psi|_{I,J} = \frac{1}{12\Delta^2}\{16[\psi(I-1,J)+\psi(I,J-1)+\psi(I+1,J)+\psi(I,J+1)] - [\psi(I,J+2)+\psi(I-2,J)+\psi(I,J-2)+\psi(I+2,J)] - 60\psi(I,J)\} + O(\Delta^4)}$$

• Elliptic PDEs in meteorology

$\nabla^2\phi = G$ ⟶ Poisson's eq. ($\nabla^2\psi = \zeta$)
$\nabla^2\phi + H\phi = G$ ⟶ Helmholtz eq.
→ elliptic boundary value problem

↳ two ways to solve
- direct solution of simultaneous eqs. (for small domain)
- relaxation method (for large domain)

• Direct method (for solving Helmholtz eq. using finite difference method)

A general outline of the method is to first write the given equation in finite difference form for each of the N vertical levels. This gives rise to a set of N linear algebraic eqs. This set of N linear algebraic eqs is then written in matrix form and solved.

$\Delta x = \Delta y = \Delta$, $i = 1(1)K$, $j = 1(1)L$

$\phi = \phi(x,y,z)$, $G = G(x,y,z)$

Helmholtz eq →

$$\boxed{\nabla_H^2\phi + \frac{\partial^2\phi}{\partial z^2} + H\phi = G}$$

∵ Finite-difference $\left(\frac{\partial^2\phi}{\partial z^2} = \frac{\phi_{k+1}-2\phi_k+\phi_{k-1}}{(\Delta z)^2}\right)$

∵ $\nabla_H = \nabla$ for convenience

Finite-difference form →

$$\frac{\nabla^2\Phi_{k+1}+2\nabla^2\Phi_k+\nabla^2\Phi_{k-1}}{4}+\frac{\Phi_{k+1}-2\Phi_k+\Phi_{k-1}}{(\Delta z)^2}+H_k\Phi_k=G_k$$

where $k=2(1)N-1$. ↳ two-dimension

For the top layer ($k=1$)

$$\frac{\nabla^2\Phi_2+2\nabla^2\Phi_1}{4}+\frac{\Phi_2-2\Phi_1}{(\Delta z)^2}+H_1\Phi_1=G_1$$

For the bottom layer ($k=N$)

$$\frac{2\nabla^2\Phi_N+\nabla^2\Phi_{N-1}}{4}+\frac{\Phi_{N-1}-2\Phi_N}{(\Delta z)^2}+H_N\Phi_N=G_N$$

where we assume $\Phi_0=\Phi_{N+1}=0$

→ matrix form

$$\nabla^2(A\Phi)+B\Phi=G^* \quad \text{——— ①}$$

where A, B : N×N coefficient matrix.

$$\Phi=\begin{pmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_N\end{pmatrix} \qquad G^*=\begin{pmatrix}G_1\\ G_2\\ \vdots\\ G_N\end{pmatrix}$$

$B^{-1}*① \to$

$$B^{-1}\nabla^2(A\Phi)+B^{-1}B\Phi=B^{-1}G^*$$

↳ $=\nabla^2(B^{-1}A\Phi)-\cancel{A\Phi\nabla^2B^{-1}}\to 0$

Since $\nabla^2B^{-1}\to B^{-1}$ is a matrix of numbers

$$\nabla^2(B^{-1}A\Phi)+\Phi=B^{-1}G^*$$

Let $B^{-1}A=C$

$$\nabla^2(C\Phi)+\Phi=B^{-1}G^*$$

One can now diagonalize matrix C using a similarity transform.

$$UCU^{-1}=D$$

↳ diagonal matrix $\begin{pmatrix}d_{11} & & 0\\ & d_{22} & \\ 0 & & d_{nn}\end{pmatrix}$

$$C=U^{-1}DU$$

↳ each d_k, $k=1(1)N$ is an eigenvalue of matrix C.

$$CX_k=d_kX_k$$ → some column vector

$$\nabla^2(U^{-1}DU\Phi)+\Phi=B^{-1}G^*$$

$$\nabla^2(DU\Phi)+U\Phi=UB^{-1}G^*$$

Let $U\Phi=V$ (a column vector)

$UB^{-1}=F$ (an N×N matrix)

$$V+\nabla^2(DV)=FG^*$$

↳ Solution will give V

$$\Phi=U^{-1}V$$

→ obtain matrix U

$CX_k=d_kX_k$ — eigenvector

$CX=XD \to C=XDX^{-1}$ ∵ $C=U^{-1}DU$

∴ $U^{-1}=X$.

• Relaxation method

Given a 5-point stencil, Laplacian of any function ϕ

$$\nabla^2\phi=\frac{\phi_{i+1,j}+\phi_{i-1,j}+\phi_{i,j+1}+\phi_{i,j-1}-4\phi_{i,j}}{\Delta^2}$$

Helmholtz eq. $\nabla^2\phi-\mu\phi=F$

$$\to \phi_{i+1,j}+\phi_{i-1,j}+\phi_{i,j+1}+\phi_{i,j-1}-[4+\mu\Delta^2]\phi_{i,j}=F_{i,j}\Delta^2 \quad \text{—— ⓐ}$$

In the relaxation method, the values of F are specified at each of the

It is desired to find the values of $\phi_{i,j}$ which satisfy ⓐ and the given boundary conditions.

* boundary conditions

① Dirichlet boundary condition
→ the value of ϕ is prescribed at the boundaries

② Neumann boundary condition
→ the value of $\partial\phi/\partial n$ is specified at the boundaries.

③ Mixed boundary condition
→ $\frac{\partial\phi}{\partial n}+\beta\phi$ is specified at the boundaries.

We assume a first guess field for ϕ, say ϕ_0, over the domain, and ask, Does ϕ_0 satisfy the Helmholtz eq. and the boundary conditions? If yes, then we have a solution. Otherwise, find the difference $\nabla^2\phi_0-\mu\phi_0-F$ → residual (R) → minimize R by iterative method.

Simultaneous relaxation scheme : uses the original values of $\phi_{i,j}$ from the previous iteration to calculate values of the next iteration.

sequential relation scheme ↳ fast
↳ uses new values of $\phi_{i,j}$ for calculating values at the next iteration

Is sequential relaxation more efficient than simultaneous relaxation?

1-D Helmholtz eq.

$$\frac{\partial^2\phi}{\partial x^2}-\mu\phi=F$$

domain $i-2,\ i-1,\ i,\ i+1,\ i+2$ — boundary → ϕ_{i-2} and ϕ_{i+2} are known

↳ $\left.\frac{\partial^2\phi}{\partial x^2}\right|_i\simeq\frac{\phi_{i+1}+\phi_{i-1}-2\phi_i}{(\Delta x)^2}$

$$\phi_{i+1}+\phi_{i-1}-2\phi_i-\mu(\Delta x)^2\phi_i=F_i(\Delta x)^2$$

Let $H=\mu(\Delta x)^2$, $(\Delta x)^2=d^2$

$$\phi_{i-1}-(2+H)\phi_i+\phi_{i+1}=F_id^2$$

→ simultaneous relaxation

at mth level of iteration

$$\phi^m_{i-1}-(2+H)\phi^m_i+\phi^m_{i+1}=F_id^2+R^m_i$$

at the (m+1)th iteration, $\phi^m_i\to\phi^{m+1}_i$ (R^m_i vanishes)

$$\phi^m_{i-1}-(2+H)\phi^{m+1}_i+\phi^m_{i+1}=F_id^2$$

Corresponding error eqs

$$\epsilon^m_{i-1}-(2+H)\epsilon^{m+1}_i+\epsilon^m_{i+1}=0 \quad \text{at grid point } i \text{ —— ⓐ}$$

$$0-(2+H)\epsilon^{m+1}_{i-1}+\epsilon^m_i=0 \quad \text{" " " } i-1 \text{ —— ⓑ}$$

$$\epsilon^m_i-(2+H)\epsilon^{m+1}_{i+1}=0 \quad \text{" " " } i+1 \text{ —— ⓒ}$$

One additional iteration at point i.

$$\epsilon^{m+1}_{i-1}-(2+H)\epsilon^{m+2}_i+\epsilon^{m+1}_{i+1}=0$$

eliminate ϵ^{m+1}_{i-1} & ϵ^{m+1}_{i+1} using ⓑ+ⓒ at grid points $(i-1)$ & $(i+1)$

$$\frac{\epsilon^m_i}{2+H}-(2+H)\epsilon^{m+2}_i+\frac{\epsilon^m_i}{2+H}=0$$

$$2\epsilon^m_i-(2+H)^2\epsilon^{m+2}_i=0$$

$$\therefore\ \epsilon^{m+2}_i=\frac{2}{(2+H)^2}\epsilon^m_i \quad \text{——— (i)}$$

→ sequential relaxation : use the new values of the function at the previously corrected points.

Corresponding error eqs

$$\epsilon^{m+1}_{i-1}-(2+H)\epsilon^{m+1}_i+\epsilon^m_{i+1}=0 \quad \text{at grid point } i \text{ —— ⓐ}$$

$$0-(2+H)\epsilon^{m+1}_{i-1}+\epsilon^m_i=0 \quad \text{" " " } i-1 \text{ —— ⓑ}$$

$$\epsilon^{m+1}_i-(2+H)\epsilon^{m+1}_{i+1}+0=0 \quad \text{" " " } i+1 \text{ —— ⓒ}$$

$\text{ⓑ}' \to \epsilon_{i-1}^{m+1} = \dfrac{\epsilon_i^m}{2+H}$

$\text{ⓐ}' \to$

$$\frac{\epsilon_i^m}{2+H} - (2+H)\epsilon_i^{m+1} + \epsilon_{i+1}^m = 0 \qquad \text{ⓒ}\ \epsilon_{i+1}^m = \frac{\epsilon_i^m}{2+H}$$

or

$$\epsilon_i^m - (2+H)^2\epsilon_i^{m+1} + (2+H)\epsilon_{i+1}^m = 0$$

$$\epsilon_i^m - (2+H)^2\epsilon_i^{m+1} + \epsilon_i^m = 0$$

$$\therefore\ \epsilon_i^{m+1} = \frac{2}{(2+H)^2}\epsilon_i^m \quad \text{—— ⓘⓘ}$$

Thus, the convergence given by ⓘⓘ is twice as fast as that given by ⓘ.

$\therefore$ Sequential relaxation converges faster than simultaneous relaxation

⊙ Barotropic vorticity eq.

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - g\frac{\partial Z}{\partial x} \quad \text{ⓐ}$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - g\frac{\partial Z}{\partial x} \quad \text{ⓑ}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

unknowns, u, v, Z.

For nondivergent flow $u = -\dfrac{\partial \psi}{\partial y}$, $v = \dfrac{\partial \psi}{\partial x}$, $\nabla^2\psi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = \zeta$

streamfunction (rotational part of flow)

→ barotropic vorticity eq.

$$\frac{\partial}{\partial t}\nabla^2\psi = -J(\psi, \nabla^2\psi) - \beta\frac{\partial \psi}{\partial x} \quad ① \quad \text{where } \beta = \frac{\partial f}{\partial y}$$

→ nonlinear balanced eq. (divergent eq.)

$$\nabla^2\phi = \nabla\cdot f\nabla\psi + 2J\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) \quad ②$$

2 eqs, 2 unknowns (ψ, ϕ)

where $J(A,B) = \dfrac{\partial A}{\partial x}\dfrac{\partial B}{\partial y} - \dfrac{\partial A}{\partial y}\dfrac{\partial B}{\partial x}$: Jacobian

* $-J(\psi, A)$: A is being advected along the isopleths of ψ

↳ "nonlinear advection term"

※ derive the barotropic vorticity eq. ①

$$\frac{\partial}{\partial x}\text{ⓑ} \to \frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} = -f\frac{\partial u}{\partial x} - g\frac{\partial^2 Z}{\partial x \partial y}$$

$$-\Big)\ \frac{\partial}{\partial y}\text{ⓐ} \to \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = f\frac{\partial v}{\partial y} + v\frac{\partial f}{\partial y} - g\frac{\partial^2 Z}{\partial x \partial y}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -f\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} - v\beta$$

$$\therefore\ \frac{\partial}{\partial t}\nabla^2\psi - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = -\beta\frac{\partial \psi}{\partial x}$$

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \beta\frac{\partial \psi}{\partial x}$$

$$= -\left(\frac{\partial \psi}{\partial x}\frac{\partial \nabla^2\psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2\psi}{\partial x}\right) - \beta\frac{\partial \psi}{\partial x}$$

$$= -J(\psi, \nabla^2\psi) - \beta\frac{\partial \psi}{\partial x}$$

This eq. can be rewritten as

$$\frac{\partial}{\partial t}(\nabla^2\psi + f) = -J(\psi, \nabla^2\psi + f)$$

※ derive divergent eq. ② $\quad \because \phi = gz$

$$\frac{\partial}{\partial x}\text{ⓐ} \to \frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial}{\partial t}\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x}\right)^2 - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - v\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial}{\partial x}(fv)$$

$$+\Big)\ \frac{\partial}{\partial y}\text{ⓑ} \to \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial}{\partial t}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial x \partial y} - \left(\frac{\partial v}{\partial y}\right)^2 - v\frac{\partial^2 v}{\partial y^2} - \frac{\partial}{\partial y}(fu)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial}{\partial t}\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} - \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 - 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - u\frac{\partial}{\partial x}\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}$$

$$- v\frac{\partial}{\partial y}\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \frac{\partial}{\partial x}(fv) - \frac{\partial}{\partial y}(fu)$$

$$\nabla^2\phi = \frac{\partial}{\partial x}\left(f\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial \psi}{\partial y}\right) - \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 + 2\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2}$$

$$= \nabla\cdot f\nabla\psi + 2\left(\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2\right)$$

$$= \nabla\cdot f\nabla\psi + 2\left(\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)\right)$$

$$= \nabla\cdot f\nabla\psi + 2J\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right)$$

⇒ Parcel invariant : $\dfrac{d\zeta_a}{dt} = 0$ ← absolute vorticity $\zeta_a = \zeta + f$

⇒ Domain invariant

$$\begin{cases} \overline{\overline{K}}^{xy} = \text{const.} & \text{(kinetic energy)} \\ \overline{\overline{\zeta_a}} = \text{const.} & \\ \overline{\overline{\zeta_a^{(n)}}} = \text{const.} & \text{→ can be any number (real)} \end{cases}$$

• Let's prove $\overline{\overline{\zeta_a^2}} = \text{const.}$

$$\frac{\partial}{\partial t}(\nabla^2\psi + f) = -J(\psi, \nabla^2\psi + f) \quad \text{—— ⓘ}$$

$(\nabla^2\psi + f) * \text{ⓘ} \to$

$$\frac{\partial}{\partial t}\frac{(\nabla^2\psi + f)^2}{2} = -J\left(\psi, \frac{(\nabla^2\psi + f)^2}{2}\right)$$

Integrating over a closed domain D,

$$\iint_D \frac{\partial}{\partial t}\frac{(\nabla^2\psi + f)^2}{2}dxdy = -\iint_D J\left(\psi, \frac{(\nabla^2\psi + f)^2}{2}\right)dxdy = 0$$

Since the integral of a Jacobian over a closed domain vanishes.

* Closed domain $\begin{cases} \overline{\overline{J(\psi, A)}} = 0 \\ \overline{\overline{\nabla\cdot A}} = 0 \end{cases}$

$$\therefore\ \overline{\overline{\frac{(\nabla^2\psi + f)^2}{2}}} = \text{const.} \Rightarrow \overline{\overline{\zeta_a^2}} = \text{const.}$$

enstrophy (square of the vorticity)

• Let's prove $\overline{\overline{K}} = \overline{\overline{\dfrac{(u^2+v^2)}{2}}} = \text{const.}$

$\psi * \text{ⓘ} \to$

$$\psi\frac{\partial}{\partial t}\nabla^2\psi = -J\left(\frac{\psi^2}{2}, \nabla^2\psi + f\right)$$

$$\iint_D \psi\frac{\partial}{\partial t}\nabla^2\psi\, dxdy = 0.$$

$\because$ vector identity $\quad$ ※ $\nabla\cdot(\phi A) = \phi\nabla\cdot A + A\cdot\nabla\phi$

$$\psi\frac{\partial}{\partial t}\nabla^2\psi = \nabla\cdot\left(\psi\nabla\frac{\partial \psi}{\partial t}\right) - \frac{\partial}{\partial t}\frac{\nabla\psi\cdot\nabla\psi}{2}$$

$$\iint_D \psi\frac{\partial}{\partial t}\nabla^2\psi\, dxdy = \cancel{\iint_D \nabla\cdot\left(\psi\nabla\frac{\partial \psi}{\partial t}\right)dxdy}^{\,0} - \iint_D \frac{\partial}{\partial t}\frac{\nabla\psi\cdot\nabla\psi}{2}$$

$$\iint_D \frac{\partial}{\partial t}\frac{(\nabla\psi)^2}{2}dxdy = \iint_D \psi\frac{\partial}{\partial t}\nabla^2\psi\, dxdy = 0$$

$$\therefore\ \frac{\partial}{\partial t}\overline{\overline{\frac{(\nabla\psi)^2}{2}}} = \frac{\partial}{\partial t}\overline{\overline{\frac{u^2+v^2}{2}}} = 0$$

$\because \nabla\psi\cdot\nabla\psi = \left(\frac{\partial \psi}{\partial x}i + \frac{\partial \psi}{\partial y}j\right)\left(\frac{\partial \psi}{\partial x}i + \frac{\partial \psi}{\partial y}j\right)$
$= (vi - uj)(vi - uj)$
$= v^2 + u^2$

※ another method

$$u * \text{ⓐ} \to \frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + u\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + v\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) = fuv - u\frac{\partial \phi}{\partial x}$$

$$+\Big)\ v * \text{ⓑ} \to \frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) + u\frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) + v\frac{\partial}{\partial y}\left(\frac{v^2}{2}\right) = -fuv - v\frac{\partial \phi}{\partial y}$$

$$\frac{\partial K}{\partial t} + u\frac{\partial K}{\partial x} + v\frac{\partial K}{\partial y} = -u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} \qquad \text{where } K = \frac{u^2+v^2}{2}$$

$$\frac{\partial K}{\partial t} = -\frac{\partial Ku}{\partial x} - \frac{\partial Kv}{\partial y} + K\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}^{\,0} - \frac{\partial \phi u}{\partial x} - \frac{\partial \phi v}{\partial y} + \phi\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}^{\,0}$$

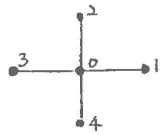
⟶ flux form

$$\frac{\partial K}{\partial t} = -\nabla\cdot(K\mathbf{V}) - \nabla\cdot(\phi\mathbf{V})$$

$$\iint_D \frac{\partial K}{\partial t}dxdy = -\cancel{\iint_D \nabla\cdot(K\mathbf{V})dxdy}^{\,0} - \cancel{\iint_D \nabla\cdot(\phi\mathbf{V})dxdy}^{\,0} = 0$$

$$\therefore\ \frac{\partial \overline{\overline{K}}}{\partial t} = 0 \qquad \overline{\overline{K}} = \text{const.}$$

• The 5-point Jacobian



$$J(\zeta,\psi) = \frac{\partial\zeta}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\zeta}{\partial y}\frac{\partial\psi}{\partial x}$$

$$J(\zeta,\psi) \cong \frac{\zeta_1-\zeta_3}{2\Delta x}\,\frac{\psi_2-\psi_4}{2\Delta y} - \frac{\zeta_2-\zeta_4}{2\Delta y}\,\frac{\psi_1-\psi_3}{2\Delta x}$$

If $\Delta x = \Delta y = \Delta$, then

$$J(\zeta,\psi) = \frac{1}{4\Delta^2}\left[(\zeta_1-\zeta_3)(\psi_2-\psi_4) - (\zeta_2-\zeta_4)(\psi_1-\psi_3)\right] + O(\Delta^2)$$

↳ a second-order accurate Jacobian

① Arakawa Jacobian

* The 5-point Jacobian conserves $\overline{\zeta_a}$, but not $\overline{K}$ or $\overline{\zeta_a^2}$.
Non-conservation of $\overline{K}$ and $\overline{\zeta_a^2}$ results in nonlinear instability in a numerical model.

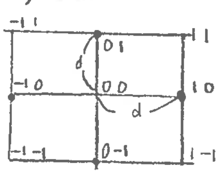
Three forms of $J(\zeta,\psi)$

$$J(\zeta,\psi) = \frac{\partial\zeta}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\zeta}{\partial y}\frac{\partial\psi}{\partial x} \quad \text{— ①}$$

$$J(\zeta,\psi) = \frac{\partial}{\partial x}\left(\zeta\frac{\partial\psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\zeta\frac{\partial\psi}{\partial x}\right) \quad \text{— ②}$$

$$J(\zeta,\psi) = \frac{\partial}{\partial y}\left(\psi\frac{\partial\zeta}{\partial x}\right) - \frac{\partial}{\partial x}\left(\psi\frac{\partial\zeta}{\partial y}\right) \quad \text{— ③}$$

→ analytically same, not in finite-difference form



9-point stencil

$\circlearrowleft 45°$

$$J_{00}^{++}(\zeta,\psi) = \frac{1}{4d^2}\left[(\zeta_{10}-\zeta_{-10})(\psi_{01}-\psi_{0-1}) - (\zeta_{01}-\zeta_{0-1})(\psi_{10}-\psi_{-10})\right] \quad \text{— ①′}$$

$$J_{00}^{+\times}(\zeta,\psi) = \frac{1}{4d^2}\left[\zeta_{10}(\psi_{11}-\psi_{1-1}) - \zeta_{-10}(\psi_{-11}-\psi_{-1-1}) - \zeta_{01}(\psi_{11}-\psi_{-11}) + \zeta_{0-1}(\psi_{1-1}-\psi_{-1-1})\right] \quad \text{— ②′}$$

$$J_{00}^{\times+}(\zeta,\psi) = \frac{1}{4d^2}\left[\psi_{01}(\zeta_{11}-\zeta_{-11}) - \psi_{0-1}(\zeta_{1-1}-\zeta_{-1-1}) - \psi_{10}(\zeta_{11}-\zeta_{1-1}) + \psi_{-10}(\zeta_{-11}-\zeta_{-1-1})\right]$$
$$= \frac{1}{4d^2}\left[\zeta_{11}(\psi_{01}-\psi_{10}) - \zeta_{-1-1}(\psi_{-10}-\psi_{0-1}) - \zeta_{-11}(\psi_{01}-\psi_{-10}) + \zeta_{1-1}(\psi_{10}-\psi_{0-1})\right] \quad \text{— ③′}$$

the conservation of mean-square vorticity

$$\overline{\zeta J(\zeta,\psi)} = 0$$

①′ →

$$\zeta_{00}J_{00}^{++}(\zeta,\psi) = \frac{1}{4d^2}\left[\zeta_{00}\zeta_{10}(\psi_{01}-\psi_{0-1}) + \cdots\right]$$
$$\zeta_{10}J_{10}^{++}(\zeta,\psi) = \frac{1}{4d^2}\left[\zeta_{10}\zeta_{00}(\psi_{11}-\psi_{1-1}) + \cdots\right]$$

↳ does not vanish

②′ →

$$\zeta_{00}J_{00}^{+\times}(\zeta,\psi) = \frac{1}{4d^2}\left[\zeta_{00}\zeta_{10}(\psi_{11}-\psi_{1-1}) + \cdots\right]$$
$$\zeta_{10}J_{10}^{+\times}(\zeta,\psi) = \frac{1}{4d^2}\left[-\zeta_{10}\zeta_{00}(\psi_{01}-\psi_{0-1}) + \cdots\right]$$

③′ →

$$\zeta_{00}J_{00}^{\times+}(\zeta,\psi) = \frac{1}{4d^2}\left[\zeta_{00}\zeta_{11}(\psi_{01}-\psi_{10}) + \cdots\right]$$
$$\zeta_{11}J_{11}^{\times+}(\zeta,\psi) = \frac{1}{4d^2}\left[-\zeta_{11}\zeta_{00}(\psi_{01}-\psi_{10}) + \cdots\right]$$

∴ $\overline{\zeta J^{++}(\zeta,\psi)}$ does not vanish.

$\overline{\zeta J^{+\times}(\zeta,\psi)}$ " " "

however,

$$\overline{\zeta J^{++}(\zeta,\psi)} + \overline{\zeta J^{+\times}(\zeta,\psi)} = 0$$

also, $\overline{\zeta J^{\times+}(\zeta,\psi)} = 0$

Similarly,

$$\overline{\psi J^{++}(\zeta,\psi)} + \overline{\psi J^{\times+}(\zeta,\psi)} = 0 \quad \leftarrow \text{K.E}$$

$$\overline{\psi J^{+\times}(\zeta,\psi)} = 0$$

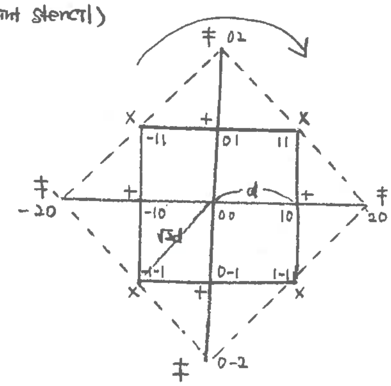
$$\therefore J_1(\zeta,\psi) = \frac{1}{3}\left[J^{++}(\zeta,\psi) + J^{+\times}(\zeta,\psi) + J^{\times+}(\zeta,\psi)\right]$$

↳ conserves $\overline{\zeta^2}$, $\overline{K}$ ↳ second order accurate.

After expand ζ & ψ in Taylor series, substitute these into ①, ②, ③,

$$J_1(\zeta,\psi) = J(\zeta,\psi) + \varepsilon d^2 + O(d^4) \quad \text{— (I)}$$

→ another form of Jacobian which conserves ζ^2 + K
(13-point stencil)



→ 45° rotation of the previous one

$$J_2(\zeta,\psi) = \frac{1}{3}\left[J^{\times\times}(\zeta,\psi) + J^{\times\ddagger}(\zeta,\psi) + J^{\ddagger\times}(\zeta,\psi)\right]$$

↳ conserves $\overline{\zeta^2}$, $\overline{K}$ ↳ second order accurate

where

$$J_{00}^{\times\times}(\zeta,\psi) = \frac{1}{8d^2}\left[(\zeta_{11}-\zeta_{-1-1})(\psi_{-11}-\psi_{1-1}) - (\zeta_{-11}-\zeta_{1-1})(\psi_{11}-\psi_{-1-1})\right]$$

$$J_{00}^{\times\ddagger}(\zeta,\psi) = \frac{1}{8d^2}\left[\zeta_{11}(\psi_{02}-\psi_{20}) - \zeta_{-1-1}(\psi_{-20}-\psi_{0-2}) - \zeta_{-11}(\psi_{02}-\psi_{-20}) + \zeta_{1-1}(\psi_{20}-\psi_{0-2})\right]$$

$$J_{00}^{\ddagger\times}(\zeta,\psi) = \frac{1}{8d^2}\left[\psi_{-11}(\zeta_{02}-\zeta_{-20}) - \psi_{1-1}(\zeta_{20}-\zeta_{0-2}) - \psi_{11}(\zeta_{02}-\zeta_{20}) + \psi_{-1-1}(\zeta_{-20}-\zeta_{0-2})\right]$$
$$= \frac{1}{8d^2}\left[\zeta_{20}(\psi_{11}-\psi_{1-1}) - \zeta_{-20}(\psi_{-11}-\psi_{-1-1}) - \zeta_{02}(\psi_{11}-\psi_{-11}) + \zeta_{0-2}(\psi_{1-1}-\psi_{-1-1})\right]$$

Using Taylor series,

$$J_2 = \frac{1}{3}\left[J^{\times\times}(\zeta,\psi) + J^{\times\ddagger}(\zeta,\psi) + J^{\ddagger\times}(\zeta,\psi)\right]$$
$$= J(\zeta,\psi) + 2\varepsilon d^2 + O(d^4) \quad \text{— (II)}$$

(I), (II) →

$$J_3(\zeta,\psi) = 2J_1(\zeta,\psi) - J_2(\zeta,\psi) = J(\zeta,\psi) + O(d^4)$$

↳ fourth-order accurate Arakawa Jacobian
→ conserve $\overline{K}$, $\overline{\zeta^2}$

Time-differencing Schemes

• amplification factor (λ)

↳ magnitude would determine whether a scheme is stable or not.

linear wave eq.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad \text{(PDE)} \longrightarrow \text{Let analytic solution } u(x,t) = \mathrm{Re}\left[U(t)e^{ikx}\right]$$

$$\frac{dU}{dt} + ikcU = 0 \quad \text{(ODE)}$$

↳ $U(t) = U(0)e^{-ikct}$

∴ harmonic solution → $u(x,t) = \mathrm{Re}\left[U(0)e^{ik(x-ct)}\right]$

→ finite-difference form

$U_j^n = \mathrm{Re}[U^n e^{ikj\Delta x}]$ n : time level

$U^n \equiv \lambda U^{n-1}$ or $|U^n| = |\lambda||U^{n-1}|$

$|U^{n-1}| = |\lambda||U^{n-2}|$, $|U^{n-2}| = |\lambda||U^{n-3}|$

$\therefore |U^n| = |\lambda|^n |U^0|$ → from this $|\lambda| \le 1$ in order for $|U^n|$ to be stable.

To be stable,

$|U^n| = |\lambda|^n |U^0| < A$ ← some finite number

$\ln|U^n| = n\ln|\lambda| + \ln|U^0| < \ln A$

$n \ln|\lambda| < \ln\frac{A}{|U^0|} \to A'$

Let $t = n\Delta t$

$\ln|\lambda| < \frac{A'}{n} \longrightarrow \ln|\lambda| < \frac{A'}{t}\Delta t$

$\therefore \ln|\lambda| \le O(\Delta t)$

If $|\lambda| \equiv 1+\alpha$, then

$\ln(1+\alpha) = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \frac{\alpha^4}{4} + \cdots$ for $-1 \le \alpha \le 1$

$\therefore |\lambda| \le 1 + O(\Delta t)$

↳ ~~von Neumann constant condition for stability~~

von Neumann necessary condition for stability

$|\lambda| \le 1$: sufficient condition

- $|\lambda| > 1 \longrightarrow$ unstable scheme
- $|\lambda| = 1 \longrightarrow$ neutral scheme
- $|\lambda| < 1 \longrightarrow$ stable scheme

• Euler, Backward + Trapezoidal scheme

From the linear wave eq. →

$* \frac{dU}{dt} = f(U,t)$, $U = U(t)$

where $f = -ikcU = i\omega U$, $\omega = -kc$

$U^{n+1} = U^n + \int_{n\Delta t}^{(n+1)\Delta t} f(U,t)\,dt$

→ two possible values for $f(U,t)$ → constant

(a) $f = f^n(U^n, n\Delta t) \longrightarrow$ explicit

(b) $f = f^{n+1}(U^{n+1}, (n+1)\Delta t) \longrightarrow$ implicit

Let $f(U,t) = \alpha f^n + \beta f^{n+1}$, $\alpha+\beta = 1$

$U^{n+1} = U^n + \int_{n\Delta t}^{(n+1)\Delta t} (\alpha f^n + \beta f^{n+1})\,dt$

⇒ $U^{n+1} = U^n + \Delta t(\alpha f^n + \beta f^{n+1})$

→ Euler's forward scheme ; $\alpha = 1$ + $\beta = 0$

backward scheme : $\alpha = 0$ + $\beta = 1$

trapezoidal scheme : $\alpha = \beta = 0.5$

Let's check stability

$f^n = i\omega U^n$, $f^{n+1} = i\omega U^{n+1}$

$U^{n+1} = U^n + \Delta t(\alpha i\omega U^n + \beta i\omega U^{n+1})$

$U^{n+1} = U^n \underbrace{\boxed{\frac{1+i\alpha\omega\Delta t}{1-i\beta\omega\Delta t}}}_{\lambda}$

Let $p = \omega\Delta t$

$\lambda = \frac{1+i\alpha p}{1-i\beta p} \times \frac{(1+i\beta p)}{(1+i\beta p)}$

$\lambda = \frac{1-\alpha\beta p^2 + ip(\overbrace{\alpha+\beta}^{1})}{1+\beta^2 p^2}$

$\lambda = \frac{1-\alpha\beta p^2 + ip}{1+\beta^2 p^2}$ ———— ⊛

✓ $\boxed{\text{Euler foreward Scheme}}$: $\alpha = 1$, $\beta = 0$ → always unstable!

⊛ → $\lambda = 1 + ip$

$|\lambda| = (1+p^2)^{1/2}$

Since $p = \omega\Delta t$ is real + $p^2 > 0$, $\therefore |\lambda| > 1$

✓ $\boxed{\text{Backward Scheme}}$ $\alpha = 0$, $\beta = 1$ → unconditionally stable! (damping)

⊛ → $\lambda = \frac{1+ip}{1+p^2}$

$|\lambda| = \frac{(1+p^2)^{1/2}}{1+p^2} = \frac{1}{(1+p^2)^{1/2}} < 1$.

→ The damping property of the backward scheme is desirable to reduce the amplitude of such high-frequency modes and to filter them out.

✓ $\boxed{\text{Trapezoidal Scheme}}$ $\alpha = \beta = 0.5$ → always neutral.

⊛ → $\lambda = \frac{4-p^2+i4p}{4+p^2}$

$|\lambda| = \frac{(16+p^4-8p^2+16p^2)^{1/2}}{4+p^2} = \frac{(16+8p^2+p^4)^{1/2}}{4+p^2} = 1$.

• Predictor - Corrector schemes

The step for finding $U^{(n+1)*}$ is the predictor step and the step for finding $U^{(n+1)}$ using $U^{(n+1)*}$ is the correct step

$U^{(n+1)*} = U^n + \Delta t f^n$ (predictor step)

$U^{(n+1)} = U^n + \Delta t(\alpha f^n + \beta f^{(n+1)*})$ (corrector step)

where $\alpha+\beta = 1$.

$f^{(n+1)*} = f(U^{(n+1)*}, (n+1)\Delta t)$

$f^n = f(U^n, n\Delta t)$

→ Stability

$U^{(n+1)*} = U^n + i\omega\Delta t U^n$

$U^{(n+1)} = U^n + i\omega\Delta t[\alpha U^n + \beta U^{(n+1)*}]$

$U^{(n+1)} = U^n + i\omega\Delta t[\alpha U^n + \beta(U^n + i\omega\Delta t U^n)]$

$U^{(n+1)} = [1 - \beta\omega^2\Delta t^2 + i\omega(\alpha+\beta)\Delta t]U^n$

$\lambda = \frac{U^{(n+1)}}{U^n} = (1-\beta\omega^2\Delta t^2) + i\omega(\underbrace{\alpha+\beta}_{1})\Delta t$

$\therefore |\lambda| = [(1-\beta\omega^2\Delta t^2)^2 + \omega^2\Delta t^2]^{1/2}$

✓ $\boxed{\text{Matsuno Scheme}}$ $\alpha = 0$, $\beta = 1$

If $\omega\Delta t \le 1$, → $\omega^4(\Delta t)^4 \le \omega^2(\Delta t)^2 \le 1$

$\therefore$ if $\omega\Delta t \le 1$, then $|\lambda| \le 1$ ——→ conditionally stable.

✓ $\boxed{\text{Heun's Scheme}}$ $\alpha = \beta = \frac{1}{2}$ → unstable scheme.

$\lambda = (1 - \frac{\omega^2\Delta t^2}{2}) + i\omega\Delta t$

$|\lambda| = (1 + \frac{\omega^4\Delta t^4}{4})^{1/2}$

for any $\omega\Delta t$, $\frac{\omega^4\Delta t^4}{4} > 0$

$\therefore |\lambda| > 1$

• Centered or ✓ leap-frog scheme

$U^{n+1} = U^{n-1} + \int_{(n-1)\Delta t}^{(n+1)\Delta t} f(U,t)\,dt$ ← constant

$= U^{n-1} + 2\Delta t f^n \longrightarrow O(\Delta t)^2$ ← truncation error

linear wave eq

$\frac{dU}{dt} - i\omega U = 0$ where $\omega = -kc$

→ centered time differencing

$$U^{n+1} = U^{n-1} + i\,2\omega\Delta t\,U^n$$

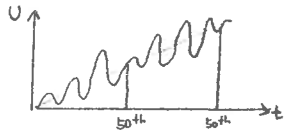
$$\because \left(U^{n+1} = \lambda U^n = \lambda^2 U^{n-1}\right)$$

$$\lambda^2 - i2p\lambda - 1 = 0 \qquad \text{where } p = \omega\Delta t$$

$$\rightarrow \begin{cases} \lambda_1 = (1-p^2)^{1/2} + ip & : \text{physical mode} \\ \lambda_2 = -(1-p^2)^{1/2} + ip & : \text{computational mode} \end{cases}$$

If $U^{n+1} = \lambda U^n$ is to represent an approx. to the true solution, $\lambda \to 1$ as $\Delta t \to 0$.

For $\begin{cases}\lambda_1, & p = \omega\Delta t \to 0 \Rightarrow \lambda_1 \to 1 \\ \lambda_2, & \quad " \quad \Rightarrow \lambda_2 \to -1\end{cases}$



To minimize the oscillation, some model uses a forward difference every 50 time step.

→ Let's examine the scheme using centered differencing both time and space.

$x = m\Delta x$, $t = n\Delta t$

$u(x,t) = U(t)e^{ikx}$

$\hookrightarrow u(m\Delta x, n\Delta t) = U(n\Delta t)e^{ikm\Delta x} = U^n e^{ikm\Delta x}$

linear wave eq →

$$U^{n+1} = U^{n-1} - \frac{c\Delta t}{\Delta x}U^n\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

$$(\because U^{n+1} = \lambda U^n = \lambda^2 U^{n-1})$$

$$\lambda^2 + i2c\frac{\Delta t}{\Delta x}\sin k\Delta x\,\lambda - 1 = 0$$

$$\hookrightarrow \begin{cases}\lambda_1 = \left[1 - \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x\right]^{1/2} - ic\frac{\Delta t}{\Delta x}\sin k\Delta x & : \text{P.M.} \\ \lambda_2 = -\left[1 - \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x\right]^{1/2} - ic\frac{\Delta t}{\Delta x}\sin k\Delta x & : \text{C.M.}\end{cases}$$

$$|\lambda_1| = |\lambda_2| = \left[1 - \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x\right]^{1/2} = 1$$

assume $U^n = \hat{u}e^{i\nu n\Delta t}$,

$$e^{i\nu\Delta t} - e^{-i\nu\Delta t} = -\frac{c\Delta t}{\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

$$\sin\nu\Delta t = -\frac{c\Delta t}{\Delta x}\sin k\Delta x$$

$\because |\sin k\Delta x| \le 1$

$\hookrightarrow$ must be $\frac{c\Delta t}{\Delta x} \le 1$ for real value of ν

$\hookrightarrow$ CFL criterion

a necessary condition for the stability of an explicit time-differencing scheme

• Adams - Bashforth Scheme

(Suitable for short periods of integration with a small time step)

$$U^{n+1} = U^n + \Delta t\left(\frac{3}{2}f^n - \frac{1}{2}f^{n-1}\right) \qquad \because f = -ikcU$$

$$U^{n+1} = U^n - ikc\Delta t\left(\frac{3}{2}U^n - \frac{1}{2}U^{n-1}\right)$$

$$\because (U^{n+1} = \lambda U^n,\ U^n = \lambda U^{n-1})$$

$$\lambda^2 = \lambda - ick\Delta t\left(\frac{3}{2}\lambda - \frac{1}{2}\right)$$

$$\lambda^2 - \left(1 - i\frac{3}{2}\omega\Delta t\right)\lambda - i\frac{1}{2}\omega\Delta t = 0 \qquad \because \omega = kc$$

$$\begin{cases}\lambda_1 = \frac{1}{2}\left[1 - i\frac{3}{2}\omega\Delta t + \left(1 - \frac{9}{4}\omega^2(\Delta t)^2 - i\omega\Delta t\right)^{1/2}\right] \\ \lambda_2 = \frac{1}{2}\left[\quad " \quad - (\quad " \quad)^{1/2}\right]\end{cases}$$

If $\Delta t \to 0$, $\begin{cases}\lambda_1 \to 1 & : \text{Physical mode} \\ \lambda_2 \to 0 & : \text{computational mode}\end{cases}$

$\Rightarrow |\lambda_1| > 1$, $|\lambda_2| < 1$

(the computational mode in this scheme tends to dampen, which is a beneficial property. However, the physical mode tends to amplify).

• Implicit Schemes

implicit finite-difference analog of the linear wave eq.

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = -\frac{c}{2}\left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\right)$$

→ The eq. can be solved by inverting a matrix with proper B.C.

assume $u_m^n = U^n e^{ikm\Delta x}$

$$\frac{U^{n+1} - U^n}{\Delta t} = -\frac{c}{2}\,\frac{U^{n+1}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)}{2\Delta x} - \frac{c}{2}\,\frac{U^n\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)}{2\Delta x}$$

$$\because \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$U^{n+1}\left(1 + ic\frac{\Delta t}{2\Delta x}\sin k\Delta x\right) = U^n\left(1 - ic\frac{\Delta t}{2\Delta x}\sin k\Delta x\right)$$

$$\therefore \lambda = \frac{U^{n+1}}{U^n} = \frac{1 - ic\frac{\Delta t}{2\Delta x}\sin k\Delta x}{1 + ic\frac{\Delta t}{2\Delta x}\sin k\Delta x}$$

$|\lambda| = 1$ → stable.

• another way of demonstrating the stability of the implicit scheme

$$u_m^{n+1} - u_m^n = -\frac{c\Delta t}{4\Delta x}\left[u_{m+1}^{n+1} + u_{m+1}^n - (u_{m-1}^{n+1} + u_{m-1}^n)\right]$$

$$\because \begin{pmatrix} u_{m+1}^{n+1} = \hat{u}e^{ik[(m+1)\Delta x + c(n+1)\Delta t]} \\ = \hat{u}e^{ik(m\Delta x + cn\Delta t)}e^{ik(\Delta x + c\Delta t)} = u_m^n e^{ik(\Delta x + c\Delta t)}\end{pmatrix}$$

$$u_m^n\left(e^{ikc\Delta t} - 1\right) = -\frac{c\Delta t}{4\Delta x}\left(e^{ik(\Delta x + c\Delta t)} + e^{ik\Delta x} - e^{ik(-\Delta x + c\Delta t)} - e^{ik(-\Delta x)}\right)u_m^n$$

$$e^{ikc\Delta t} - 1 = -\frac{c\Delta t}{4\Delta x}\left[e^{ik\Delta x}\left(e^{ikc\Delta t} + 1\right) - e^{-ik\Delta x}\left(e^{ikc\Delta t} + 1\right)\right]$$

$$= -\frac{c\Delta t}{4\Delta x}\left[\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\left(e^{ikc\Delta t} + 1\right)\right]$$

$$\frac{e^{ikc\Delta t} - 1}{e^{ikc\Delta t} + 1} = -\frac{c\Delta t}{4\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

$$\tan\frac{kc\Delta t}{2} = -\frac{c\Delta t}{2\Delta x}\sin k\Delta x \longrightarrow \text{unconditionally stable.}$$

$-\infty < \square < \infty$ $\therefore \frac{c\Delta t}{\Delta x}$ can be any value

• Shallow-water model — Exam?

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

momentum eq

$$\begin{cases}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + \frac{\partial \phi}{\partial x} = 0 & \text{— ⓐ} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + \frac{\partial \phi}{\partial y} = 0 & \text{— ⓑ}\end{cases}$$

continuity eq

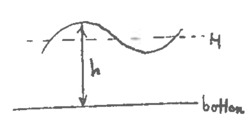
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{— ①}$$

① →

$$W_{top} - W_{bottom} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)h = 0 \qquad (\because u \,\&\, v \text{ are independent of height})$$

Let $\begin{cases} H = \text{mean depth of the fluid} \\ h' = \text{perturbation height}\end{cases}$

→ $h = H + h'$



assume $W_{bottom} = 0$ with a flat surface

$$W_{top} = \frac{dh}{dt} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - h'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - h'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad \text{— ②}$$

$g *$ ② → $(\because \phi = gh)$

$$\frac{\partial \phi'}{\partial t} + \frac{\partial}{\partial x}(u\phi') + \frac{\partial}{\partial y}(v\phi') + \Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad \text{— ⓒ}$$

where $\Phi = gH$, $\phi' = \phi - \Phi$

→ A (linearized) shallow-water system

↳ Two of the solutions are gravitational modes, and the third is a Rossby wave.

→ linearized form on a nonrotating frame ($f=0$)

$$\frac{\partial u}{\partial t}+\frac{\partial \phi'}{\partial x}=0 \quad \text{—— (I)}$$
$$\frac{\partial v}{\partial t}+\frac{\partial \phi'}{\partial y}=0 \quad \text{—— (II)}$$
$$\frac{\partial \phi'}{\partial t}+\bar{\Phi}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \quad \text{—— (III)}$$

← no mean motion

$\frac{\partial}{\partial t}$(III) and making use of (I), (II) →

$$\frac{\partial^2 \phi'}{\partial t^2}-\bar{\Phi}\,\nabla^2\phi'=0$$

Let's assume that the perturbation is only in x-direction

$$\frac{\partial^2 \phi'}{\partial t^2}-\bar{\Phi}\frac{\partial^2 \phi'}{\partial x^2}=0$$

Assuming $\phi'=e^{ik(x-ct)}$

$c^2=\bar{\Phi}=gH$ or $c=\pm\sqrt{gH}$ ⇝ 2 gravity waves.

↑ gravity wave phase speed

→ linearize eq. including the f and assuming nondivergent

$$\frac{\partial u}{\partial t}-fv=-\frac{\partial \phi'}{\partial x} \quad \text{—— (i)}$$
$$\frac{\partial v}{\partial t}+fu=-\frac{\partial \phi'}{\partial y} \quad \text{—— (ii)}$$

where $u=-\frac{\partial\psi}{\partial y}$, $v=\frac{\partial \psi}{\partial x}$, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

$\frac{\partial}{\partial x}$(ii) $-\frac{\partial}{\partial y}$(i) → (linearized form of the vorticity eq)

$$\frac{\partial \zeta}{\partial t}=-\beta v \qquad \beta=\frac{\partial f}{\partial y},\ \zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\nabla^2\psi$$

$$\frac{\partial}{\partial t}\nabla^2\psi=-\beta\frac{\partial \psi}{\partial x}$$

assume $\psi=\hat{\psi}\,e^{i(kx+ly-\nu t)}$

$$\Rightarrow \nu(k^2+l^2)=-k\beta$$
$$\nu=\frac{-k\beta}{k^2+l^2}$$
$$\Rightarrow C_x=\frac{\nu}{k}=\frac{-\beta}{k^2+l^2}$$: Rossby wave

↳ phase speed of Rossby waves

∴ we observe that the shallow-water equations contain both slow-moving Rossby waves and high-frequency gravity waves.

*<Because the nonlinear part of the eqs cannot be dealt with implicitly, we calculate it explicitly. The linear part of the eqs is treated implicitly> → a semi-implicit time-integration scheme

→ Shallow-water eqs.

$$\frac{\partial u}{\partial t}+\frac{\partial \phi'}{\partial x}=-\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv\right)=N_u \quad \text{—— (A)}$$
$$\frac{\partial v}{\partial t}+\frac{\partial \phi'}{\partial y}=-\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu\right)=N_v \quad \text{—— (B)}$$
$$\frac{\partial \phi'}{\partial t}+\bar{\Phi}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=-\left(\frac{\partial u\phi'}{\partial x}+\frac{\partial v\phi'}{\partial y}\right)=N_h \quad \text{—— (C)}$$

gravity wave terms	Rossby wave terms
(implicit) <linear>	(explicit) <non-linear>

※ Let drop the ( )′ in ϕ' term for simplicity

Finite-diff. analog of (A) →

$$\frac{u^{n+1}-u^{n-1}}{2\Delta t}+\frac{1}{2}\left(\frac{\partial\phi^{n+1}}{\partial x}+\frac{\partial \phi^{n-1}}{\partial x}\right)=(N_u)^n$$

$$u^{n+1}=u^{n-1}-\Delta t\left(\frac{\partial\phi^{n+1}}{\partial x}+\frac{\partial\phi^{n-1}}{\partial x}\right)+2\Delta t(N_u)^n \quad \text{—— (A)}'$$

similarly

$$v^{n+1}=v^{n-1}-\Delta t\left(\frac{\partial\phi^{n+1}}{\partial y}+\frac{\partial\phi^{n-1}}{\partial y}\right)+2\Delta t(N_v)^n \quad \text{—— (B)}'$$
$$\phi^{n+1}=\phi^{n-1}-\bar{\Phi}\Delta t\left[\left(\frac{\partial u^{n+1}}{\partial x}+\frac{\partial v^{n+1}}{\partial y}\right)+\left(\frac{\partial u^{n-1}}{\partial x}+\frac{\partial v^{n-1}}{\partial y}\right)\right]+2\Delta t(N_h)^n \quad \text{—— (C)}'$$

→ a single eq for ϕ $\frac{\partial}{\partial x}$(A)′, $\frac{\partial}{\partial y}$(B)′ → (C)′

$$\phi^{n+1}=\phi^{n-1}+\bar{\Phi}(\Delta t)^2\nabla^2\phi^{n+1}+\bar{\Phi}(\Delta t)^2\nabla^2\phi^{n-1}$$
$$-\bar{\Phi}\Delta t(\nabla\cdot \mathbf{V}^{n-1})-2\bar{\Phi}(\Delta t)^2\left(\frac{\partial}{\partial x}(N_u)^n+\frac{\partial}{\partial y}(N_v)^n\right)$$
$$-\bar{\Phi}\Delta t(\nabla\cdot\mathbf{V}^{n-1})+2\Delta t(N_h)^n$$

$$\bar{\Phi}(\Delta t)^2\nabla^2\phi^{n+1}-\phi^{n+1}=\underbrace{-\left[\phi^{n-1}+\bar{\Phi}(\Delta t)^2\nabla^2\phi^{n-1}-2\bar{\Phi}\Delta t(\nabla\cdot\mathbf{V}^{n-1})\right]}_{F^{n-1}}$$
$$\underbrace{+2\bar{\Phi}(\Delta t)^2\left[\frac{\partial}{\partial x}(N_u)^n+\frac{\partial}{\partial y}(N_v)^n\right]-2\Delta t(N_h)^n}_{G^n}$$

$$\Rightarrow \nabla^2\phi^{n+1}-\frac{\phi^{n+1}}{\bar{\Phi}(\Delta t)^2}=\frac{F^{n-1}+G^n}{\bar{\Phi}(\Delta t)^2}$$

↳ Helmholtz eq. for the variable ϕ^{n+1}

↳ after obtain ϕ^{n+1} $\xrightarrow{\text{substit.}}$ (A)′, (B)′ to obtain u^{n+1}, v^{n+1}.

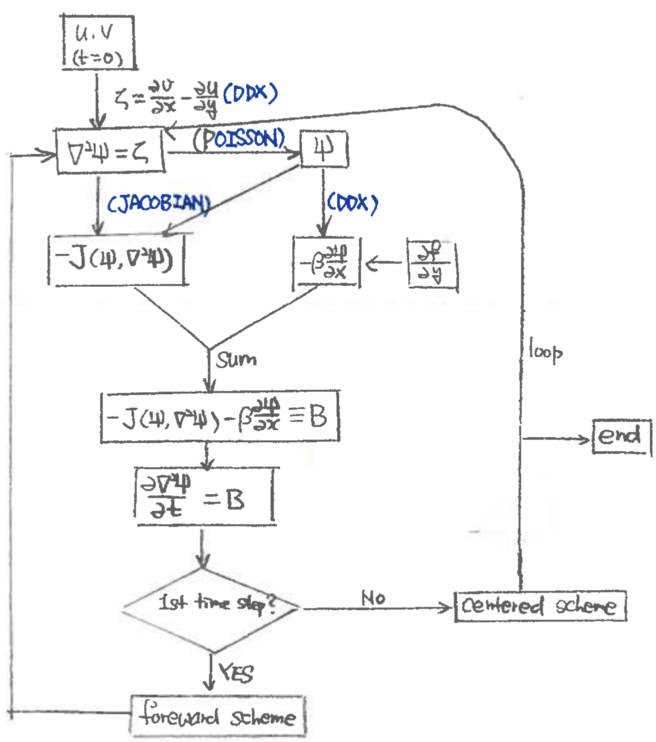
• Flow chart for barotropic vorticity eq?

$$\frac{\partial}{\partial t}\nabla^2\psi=-J(\psi,\nabla^2\psi)-\beta\frac{\partial\psi}{\partial x}\equiv B$$

First forward then on centered : time differencing.

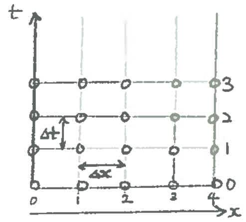
Subroutine needed ($\nabla^2\psi=\zeta$)

LAPLACIAN, JACOBIAN, DDX($\frac{\partial}{\partial x}$), MARCHING, POISSON



• What is the tridiagonal matrix for the following eq?
(implicit scheme)

$$\frac{u_m^{n+1}-u_m^{n}}{\Delta t} = -\frac{c}{2}\left(\frac{u_{m+1}^{n+1}-u_{m-1}^{n+1}}{2\Delta x} + \frac{u_{m+1}^{n}-u_{m-1}^{n}}{2\Delta x}\right) \quad \text{—— (1)}$$



Boundary Condition
$x=0 \quad u(0,t)=0$
$x=4 \quad u(4,t)=0$

← initial conditions $u(x,0)$ are knowns

rewriting eq.(1) →

$$u_m^{n+1} + \frac{c\Delta t}{4\Delta x}u_{m+1}^{n+1} - \frac{c\Delta t}{4\Delta x}u_{m-1}^{n+1} = u_m^n - \frac{c\Delta t}{4\Delta x}\left(u_{m+1}^n - u_{m-1}^n\right) \equiv \underbrace{F_m^n}_{\text{(known)}}$$

For $n=0,\ m=1$

$$u_1^1 + \frac{c\Delta t}{4\Delta x}u_2^1 - \cancel{\frac{c\Delta t}{4\Delta x}u_0^1}^{\nearrow 0\,(\mathrm{B.C})} = F_1^0$$

For $n=0,\ m=2$

$$-\frac{c\Delta t}{4\Delta x}u_1^1 + u_2^1 + \frac{c\Delta t}{4\Delta x}u_3^1 = F_2^0$$

For $n=0,\ m=3$

$$-\frac{c\Delta t}{4\Delta x}u_2^1 + u_3^1 + \cancel{\frac{c\Delta t}{4\Delta x}u_4^1}^{\nearrow 0\,(\mathrm{B.C})} = F_3^0$$

So, we can write it as a matrix form

$$\underbrace{\begin{pmatrix} 1 & \frac{c\Delta t}{4\Delta x} & 0 \\ -\frac{c\Delta t}{4\Delta x} & 1 & \frac{c\Delta t}{4\Delta x} \\ 0 & -\frac{c\Delta t}{4\Delta x} & 1 \end{pmatrix}}_{\equiv T}\begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} = \begin{pmatrix} F_1^0 \\ F_2^0 \\ F_3^0 \end{pmatrix} \quad \text{—— (2)}$$

For $n=1,2,3,\cdots$, similarly we can have (2), that is,

$$\begin{pmatrix} & & \\ & T & \\ & & \end{pmatrix}\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \end{pmatrix} = \begin{pmatrix} F_1^n \\ F_2^n \\ F_3^n \end{pmatrix}$$

This result can be applied to a more large grid point case (k)
Therefore, the general form of the tridiagonal matrix is

$$\begin{pmatrix} 1 & \frac{c\Delta t}{4\Delta x} & & & 0 \\ -\frac{c\Delta t}{4\Delta x} & 1 & \frac{c\Delta t}{4\Delta x} & & \\ & -\frac{c\Delta t}{4\Delta x} & 1 & \ddots & \\ & & \ddots & \ddots & \frac{c\Delta t}{4\Delta x} \\ 0 & & & -\frac{c\Delta t}{4\Delta x} & 1 \end{pmatrix}\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_k^{n+1} \end{pmatrix} = \begin{pmatrix} F_1^n \\ F_2^n \\ F_3^n \\ \vdots \\ F_k^n \end{pmatrix}$$

• What is "spectral modeling"?

If one takes a closed system of the basic meteorological eqs. and introduces with this system a finite expansion of the dependent variables using functions such as double Fourier or Fourier-Legendre functions in space, then the use of the orthogonality properties of these spatial functions enables one to obtain a set of coupled nonlinear ordinary differential eqs for the coefficients of these functions. These coefficients are functions of time and the vertical coordinate, since the horizontal spatial dependence has been removed by taking a Fourier or a Fourier-Legendre transform of the eqs. The coupled nonlinear ordinary differential eqs for the coefficients are usually solved by simple time-differencing and vertical finite-differencing schemes. The mapping of the solution requires the multiplication of the coefficients with the spatial functions summed over a set of chosen finite spatial basis functions.

Lower-Order Spectral model

• barotropic vorticity eq

(for 2-D, homogeneous, incompressible, inviscid fluid on an f-plane)

$$\frac{\partial}{\partial t}\nabla^2\psi = -J(\psi,\nabla^2\psi) \quad \text{or} \quad \frac{\partial}{\partial t}\nabla^2\psi = -\mathbf{k}\cdot\nabla\psi\times\nabla(\nabla^2\psi) \quad \text{—— (1)}$$

Consider the periodicity property

$$\psi(x,y,t) = \psi\Big(\underbrace{x+\frac{2\pi}{k},\ y+\frac{2\pi}{l}}_{\to\ \text{double periodic}},\ t\Big)$$

Expand ψ, (Lorenz, 1960) function of time

✓ $$\psi = \sum_{m=0}^{\infty}\sum_{n=n_0}^{\infty} \frac{-1}{m^2k^2+n^2l^2}\left[A_{mn}\cos(mkx+nly) + B_{mn}\sin(mkx+nly)\right] \quad \text{—— (2)}$$

→ double Fourier representation

where m, n : east-west, north-south wavenumbers

Note $A_{00} = 0$

$$n_0 = \begin{cases} -\infty & \text{if } m>0 \\ 0 & \text{if } m=0 \end{cases}$$

Let truncate the series, just consider

$$\left\langle \begin{matrix} m = 0, +1 \\ n = -1, 0, 1 \end{matrix} \right\rangle$$

(2)→

(3) $$\psi = -\frac{A_{10}}{k^2}\cos kx - \frac{A_{01}}{l^2}\cos ly - \frac{A_{11}}{k^2+l^2}\cos(kx+ly) - \frac{A_{1-1}}{k^2+l^2}\cos(kx-ly) - \frac{B_{10}}{k^2}\sin kx - \frac{B_{01}}{l^2}\sin ly - \frac{B_{11}}{k^2+l^2}\sin(kx+ly) - \frac{B_{1-1}}{k^2+l^2}\sin(kx-ly)$$

(4) $$\nabla^2\psi = A_{10}\cos kx + A_{01}\cos ly + A_{11}\cos(kx+ly) + A_{1-1}\cos(kx-ly) + B_{10}\sin kx + B_{01}\sin ly + B_{11}\sin(kx+ly) + B_{1-1}\sin(kx-ly)$$

⟹ maximum simplication

(3),(4) → (1), taking the Fourier transform of both sides of the resulting eq, we get predition eqs for the amplitude of the different wave components

→ 8 eqs

assume (a) If $B_{10}, B_{01}, B_{11}, B_{1-1}$ vanish initially, then

$$\frac{dB_{10}}{dt} = \frac{dB_{01}}{dt} = \frac{dB_{11}}{dt} = \frac{dB_{1-1}}{dt} = 0$$

$\therefore B_{10} = B_{01} = B_{11} = B_{1-1} = 0$

(b) $A_{1-1} = -A_{11}$.

Let $A_{01} = A,\ A_{10} = F,\ -A_{11} = G$

(3)→ $$\psi = \underbrace{-\frac{A}{l^2}\cos ly}_{\text{basic zonal current}} \underbrace{- \frac{F}{k^2}\cos kx - 2\frac{G}{k^2+l^2}\sin kx\sin ly}_{\text{eddy}} \quad : \text{mapping}$$

$$\nabla^2\psi = A\cos ly + F\cos kx + 2G\sin kx\sin ly \quad \text{—— (5)}$$

$$J(\psi,\nabla^2\psi) = \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}$$

derive

$$\therefore \begin{cases} \dfrac{\partial\psi}{\partial x} = \dfrac{F}{k}\sin kx - \dfrac{2Gk}{k^2+l^2}\cos kx\sin ly \\ \dfrac{\partial\psi}{\partial y} = \dfrac{A}{l}\sin ly - \dfrac{2Gl}{k^2+l^2}\sin kx\cos ly \\ \dfrac{\partial\nabla^2\psi}{\partial x} = -Fk\sin kx + 2Gk\cos kx\sin ly \\ \dfrac{\partial\nabla^2\psi}{\partial y} = -Al\sin ly + 2Gl\sin kx\cos ly \end{cases}$$

$$J(\psi,\nabla^2\psi) = \left(\frac{1}{l^2}-\frac{1}{k^2}\right)AFkl\sin kx\sin ly + \left(\frac{1}{k^2}-\frac{1}{k^2+l^2}\right)2FGkl\sin^2 kx\cos ly + \left(-\frac{1}{l^2}+\frac{1}{k^2+l^2}\right)2AGkl\sin^2 ly\cos kx \quad \text{—— (6)}$$

(5)→

$$\frac{\partial}{\partial t}\nabla^2\psi = \frac{dA}{dt}\cos ly + \frac{dF}{dt}\cos kx + 2\frac{dG}{dt}\sin kx\sin ly$$

∴ ① →

$$\left[\frac{dA}{dt}\cos ly + \frac{dF}{dt}\cos kx + 2\frac{dG}{dt}\sin kx \sin ly = -⑥\right] \quad ⑦$$

If we multiply the above eq by $\cos ly$ and integrate both sides over the entire doubly periodic domain and using the orthogonality properties of the Fourier function,

$$\frac{dA}{dt}\int_0^{2\pi}\int_0^{2\pi}\cos^2 ly\,dxdy$$

$$= -\left(\frac{1}{k^2}-\frac{1}{k^2+l^2}\right)2klFG \times \int_0^{2\pi}\int_0^{2\pi}\sin^2 kx\cos^2 ly\,dxdy$$

$$\therefore \frac{dA}{dt} = -\underbrace{\left(\frac{1}{k^2}-\frac{1}{k^2+l^2}\right)}_{\eta_1 = \text{known}} klFG \quad ⑧$$

Similarly, $\cos kx * ⑦$, $\sin kx \sin ly * ⑦$ → integrate

$$\frac{dF}{dt} = \underbrace{\left(\frac{1}{l^2}-\frac{1}{k^2+l^2}\right)}_{\eta_2 = \text{known}} kl AG \quad ⑨$$

$$\frac{dG}{dt} = -\frac{1}{2}\underbrace{\left(\frac{1}{l^2}-\frac{1}{k^2}\right)}_{\eta_3 = \text{known}} kl AF \quad ⑩$$

* Spectral model

Closed system of equations → transform of the closed system

Basis function → transform of the closed system

transform of the closed system → Time integration → reconstitute weather map (fcst map)

• Conservation of mean-square vorticity and mean K.E. of the above simplified system

$$\int_0^{2\pi}\int_0^{2\pi}(\nabla^2\psi)^2\,dxdy = \int_0^{2\pi}\int_0^{2\pi}(A\cos ly + F\cos kx + 2G\sin kx \sin ly)^2\,dxdy \quad ⑤$$

orthogonality →

$$= \int_0^{2\pi}\int_0^{2\pi}(A^2\cos^2 ly + F^2\cos^2 kx + 4G^2\sin^2 kx \sin^2 ly)\,dxdy$$

$$= 2\pi^2A^2 + 2\pi^2F^2 + 4\pi^2G^2$$

$* \iint dxdy = 4\pi^2$

$$= 2\pi^2(A^2+F^2+2G^2)$$

$$\therefore \overline{(\nabla^2\psi)^2} = \frac{1}{2}(A^2+F^2+2G^2)$$

$$\frac{d}{dt}\overline{(\nabla^2\psi)^2} = A\frac{dA}{dt} + F\frac{dF}{dt} + 2G\frac{dG}{dt} = 0 \quad ⑧ ⑨ ⑩$$

Thus, the mean-square vorticity is conserved.

$$\text{Total K.E.} = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right]dxdy$$

$$\overline{K.E} = \frac{1}{4}\left(\frac{F^2}{k^2} + \frac{A^2}{l^2} + \frac{2G^2}{k^2+l^2}\right)$$

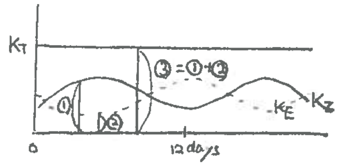
$$\frac{d}{dt}\overline{K.E} = 0$$

• Energy transformation in a barotropic model (f-plane)

$$\frac{\partial}{\partial t}\overline{K_T} = 0$$

$$\frac{\partial}{\partial t}\overline{K_Z} = \overline{u}\frac{\partial\overline{u'v'}}{\partial y}$$

$$\frac{\partial}{\partial t}\overline{K_E} = -\overline{u}\frac{\partial\overline{u'v'}}{\partial y}$$



⇒ out of phase relationship

$\langle K_E, K_Z\rangle$ & $\langle K_Z, K_E\rangle$ are equal in magnitude but opposite in sign.

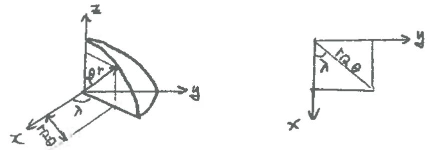
Mathematical aspects of spectral models

• An introduction to the spherical harmonics → basis function for spectral model.

Spherical harmonics are made up of trigonometric functions along the zonal direction and associated Legendre functions in the meridional direction

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad ①$$

→ The solutions of Laplace's eq. in a spherical coord. system are the spherical harmonics, and are obtained by the method of separation of variables



co-latitude : $\theta = 90° - \text{lat}$

$$x = r\cos\lambda\sin\theta, \quad y = r\sin\lambda\sin\theta, \quad z = r\cos\theta$$

① → (use a chain rule)

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\lambda^2} = 0$$

or

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\lambda^2}\right] = 0$$

$* \ \theta \xrightarrow{\text{change}}$ latitude?

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\cos^2\theta}\frac{\partial^2 u}{\partial\lambda^2} = 0$$

∵ $u = R(r)L(\lambda)P(\theta)$

* define $\mu = \sin\theta$ ∴ ∵ there is W-E part.

⟨Associated Legendre eq.⟩ (N-S part)

$$\boxed{\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + \left(n(n+1) - \frac{m^2}{1-\mu^2}\right)P = 0}$$

If $m=0$, then

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + n(n+1)P = 0 \quad \rightarrow \text{Legendre eq.}$$

→ describe only the zonally symmetric part.

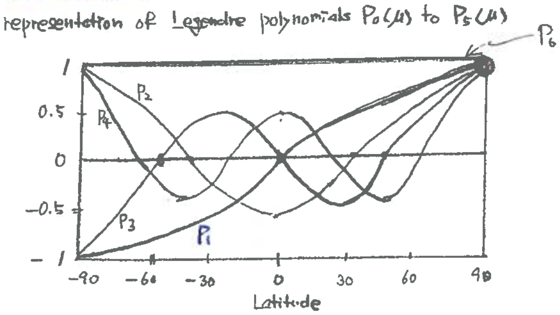
• Solutions of Legendre and associated Legendre eqs

• Solutions of Legendre eq → "Legendre polynomials" $= P_n(\mu)$

(Rodrigues' formula)

$$\boxed{P_n(\mu) = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n}, \quad n = 0, 1, 2, \cdots, \ |\mu| \le 1$$

representation of Legendre polynomials $P_0(\mu)$ to $P_5(\mu)$



Three useful properties of $P_n(\mu)$

(1) $P_n(\mu=1) = 1$

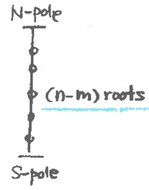
(2) If n = even, $P_n(\mu)$ has only even powers of μ (is symmetric wrt the equator ($\mu=0$))

(3) If n is odd, $P_n(\mu)$ has only odd powers of μ (is antisymmetric w.r.t. the equator)

- Solutions of the associated Legendre eq.

→ associated Legendre functions of the first kind of order m and degree n $\equiv P_n^m(\mu)$; $n \geq |m|$

$$\boxed{P_n^m(\mu) = \frac{(1-\mu^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n}$$



The $n-m$ roots between the poles are called the zeros of the associated Legendre function.

If $\begin{cases} n-m \text{ even : symmetric w.r.t the equator} \\ n-m \text{ odd : antisymmetric " "} \end{cases}$

→ useful properties

$$\begin{cases} (1)\ P_n^m(\mu) = 0 \quad \text{if } n<m \\ (2)\ P_n^{-m}(\mu) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(\mu) \ * \\ (3)\ P_n^m(-\mu) = (-1)^{n-m} P_n^m(\mu) \end{cases}$$

- Laplace eq. (spherical harmonics) continues.

Let $Y = PL$.

Laplace eq.

$$\boxed{\frac{d}{d\mu}\left((1-\mu^2)\frac{dY}{d\mu}\right) + \frac{1}{1-\mu^2}\frac{\partial^2 Y}{\partial \lambda^2} + n(n+1)Y = 0}$$

↳ form of $\nabla^2 Y + n(n+1)Y = 0$

$\therefore Y(\mu,\lambda) = P(\mu)L(\lambda)$

$$L\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + \frac{P}{1-\mu^2}\frac{\partial^2 L}{\partial\lambda^2} + n(n+1)PL = 0$$

$$\frac{1-\mu^2}{P}\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + n(n+1)(1-\mu^2) = \frac{-1}{L}\frac{d^2L}{d\lambda^2} \equiv m^2$$

R.H.S → $\frac{d^2L}{d\lambda^2} + m^2 L = 0$ → solution $L = e^{\pm im\lambda}$

L.H.S → $\frac{1-\mu^2}{P}\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + n(n+1)(1-\mu^2) = m^2$

↳ associated Legendre eq ↳ Solution P_n^m

$\therefore$ the solution of Laplace's eq. on a sphere is of the form

$$\boxed{Y_n^m(\mu,\lambda) = P_n^m(\mu)e^{im\lambda}}$$

$e^{im\lambda}$ → east-west variation; $P_n^m(\mu)$ → north-south variation; Y_n^m ↳ spherical harmonics

→ useful properties of $Y_n^m(\mu,\lambda)$

$$\begin{cases} Y_n^m(\mu,\lambda) = 0 \quad \text{for } n<m \\ Y_n^{*m}(\mu,\lambda) = P_n^m(\mu)e^{-im\lambda} \\ Y_n^{-m}(\mu,\lambda) = P_n^{-m}(\mu)e^{-im\lambda} = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(\mu)e^{-im\lambda} \\ \nabla^2 Y_n^m = \frac{-n(n+1)}{a^2}Y_n^m \ (?) \quad \nabla^2 Y_n^m = -n(n+1)Y \end{cases}$$

↳ $\nabla^2 = \frac{1}{a\cos\theta}\left(\frac{1}{\cos\theta}\frac{\partial^2}{\partial\lambda^2} + \frac{\partial}{\partial\theta}\cos\theta\frac{\partial}{\partial\theta}\right)$

- Orthogonality properties

$$\int_{-1}^{1} P_m(\mu)P_n(\mu)\,d\mu = \begin{cases} 0 & \text{if } m\neq n \\ \frac{2}{2n+1} & \text{if } m=n \end{cases}$$

$$\int_{-1}^{1} P_{n_1}^{m_1}(\mu)P_{n_2}^{m_2}(\mu)\,d\mu = \begin{cases} \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1} & \text{if } m_1=m_2=m \text{ and if } n_1=n_2=n \\ 0 & \text{if } m_1\neq m_2 \text{ and/or if } n_1\neq n_2 \end{cases}$$

In practice, for spectral modeling it is more convenient to use normalized Legendre polynomials and associated Legendre functions

$$P_n(\mu) * \left(\frac{2n+1}{2}\right)^{1/2} = \tilde{P}_n(\mu)$$

$$\therefore \int_{-1}^{1}\tilde{P}_m(\mu)\tilde{P}_n(\mu)d\mu = \begin{cases} 0 & \text{for } m\neq n \\ \dots \end{cases}$$

Likewise

$$\tilde{P}_n^m(\mu) = \left(\frac{(n-m)!}{(n+m)!}\right)^{1/2}\left(\frac{2n+1}{2}\right)^{1/2}P_n^m(\mu)$$

$$\therefore \int_{-1}^{1}\tilde{P}_{n_1}^{m_1}(\mu)\tilde{P}_{n_2}^{m_2}(\mu) = \begin{cases} 0 & \text{if } m_1\neq m_2 \text{ and/or } n_1\neq n_2 \\ 1 & \text{if } m_1=m_2=m \text{ and } n_1=n_2=n \end{cases}$$

note $\tilde{P}_n^{-m}(\mu) = (-1)^m\tilde{P}_n^m$

※ For simplicity $(\tilde{\ })$ will be droped! for the following!

- Recurrence Relations.

(1) $\mu P_n^m(\mu) = \epsilon_{n+1}^m P_{n+1}^m(\mu) + \epsilon_n^m P_{n-1}^m(\mu)$ ☆

where $\epsilon_n^m = \left(\frac{n^2-m^2}{4n^2-1}\right)^{1/2}$

(2) $(1-\mu^2)\frac{dP_n^m(\mu)}{d\mu} = -n\epsilon_{n+1}^m P_{n+1}^m(\mu) + (n+1)\epsilon_n^m P_{n-1}^m(\mu)$ ✓

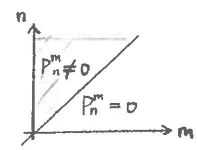
(3) $(1-\mu^2)\frac{dP_n^m(\mu)}{d\mu} = (2n+1)\epsilon_n^m P_{n-1}^m(\mu) - n\mu P_n^m(\mu)$ ☆

(4) $(1-\mu^2)^{1/2}P_n^m(\mu) = g_n^m P_{n+1}^{m+1}(\mu) - h_n^m P_{n-1}^{m+1}(\mu)$ ☆

$\because g_n^m = \left(\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}\right)^{1/2}, \quad h_n^m = \left(\frac{(n-m-1)(n-m)}{(2n+1)(2n-1)}\right)^{1/2}$

→ Starting with the value of $P_0^0(\mu)$, the above recurrence relations can generate the values of associated Legendre functions of any given order m and degree n.

$P_0^0 = 1$ → normalizing $\left(\frac{2n+1}{2}\right)^{1/2}P_0^0 = \left(\frac{1}{2}\right)^{1/2}$ → for global



$\therefore n \geq |m|$

- Gaussian Quadrature

→ Gaussian Quadrature is used for the Legendre transform of data in the north-south direction.

$n = \frac{N+1}{2} \quad (\because 2n-1 = N)$

- Spectral representation of physical fields

any smooth function over a sphere can be expressed as a sum of spherical harmonics.

- u, v → a singular behavior at the poles.

→ Robert function

$$\boxed{U = \frac{u\cos\theta}{a} \quad + \quad V = \frac{v\cos\theta}{a}}$$

→ at poles $(\theta = \pm\frac{\pi}{2})$, $U = V = 0$

$$\begin{cases} U = \frac{1}{a^2}\left(\frac{\partial\chi}{\partial\lambda} - \cos\theta\frac{\partial\psi}{\partial\theta}\right) \text{ — ①} \quad \chi\text{: velocity potential} \\ V = \frac{1}{a^2}\left(\frac{\partial\psi}{\partial\lambda} + \cos\theta\frac{\partial\chi}{\partial\theta}\right) \text{ — ②} \\ \zeta = \frac{1}{\cos^2\theta}\left(\frac{\partial V}{\partial\lambda} - \cos\theta\frac{\partial U}{\partial\theta}\right) \text{ — ③} \\ D = \frac{1}{\cos^2\theta}\left(\frac{\partial U}{\partial\lambda} + \cos\theta\frac{\partial V}{\partial\theta}\right) \text{ — ④} \end{cases}$$

$$\begin{cases} \psi = a^2\sum_m\sum_n \psi_n^m Y_n^m(\lambda,\mu) \text{ — ⑤} \\ \chi = a^2\sum_m\sum_n \chi_n^m Y_n^m(\lambda,\mu) \text{ — ⑥} \end{cases}$$

$$\begin{cases} \zeta = \nabla^2\psi = \sum_m\sum_n -n(n+1)\psi_n^m Y_n^m \\ D = \nabla^2\chi = \sum_m\sum_n -n(n+1)\chi_n^m Y_n^m \end{cases}$$

$$\therefore \begin{cases} \zeta_n^m = -n(n+1)\psi_n^m \\ D_n^m = -n(n+1)\chi_n^m \end{cases}$$

⑤,⑥ → ①,②

and using $\frac{\partial Y_n^m(\lambda,\mu)}{\partial \lambda} = im Y_n^m(\lambda,\mu)$

recurrence relations (2)

and $(1-\mu^2)\frac{\partial Y_n^m(\lambda,\mu)}{\partial \mu} = -n\epsilon_{n+1}^m Y_{n+1}^m(\lambda,\mu) + (n+1)\epsilon_n^m Y_{n-1}^m(\lambda,\mu)$

$\Rightarrow$

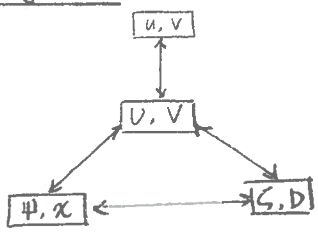
$$U = \sum_m\sum_n U_n^m Y_n^m = \sum_m\sum_n im\chi_n^m Y_n^m - \sum_m\sum_n \psi_n^m\left[-n\epsilon_{n+1}^m Y_{n+1}^m + (n+1)\epsilon_n^m Y_{n-1}^m\right]$$

$$V = \sum_m\sum_n V_n^m Y_n^m = \sum_m\sum_n im\psi_n^m Y_n^m + \sum_m\sum_n \chi_n^m\left[-n\epsilon_{n+1}^m Y_{n+1}^m + (n+1)\epsilon_n^m Y_{n-1}^m\right]$$

$$\therefore \begin{cases} U_n^m = im\chi_n^m + (n-1)\epsilon_n^m\psi_{n-1}^m - (n+2)\epsilon_{n+1}^m\psi_{n+1}^m \\ V_n^m = im\psi_n^m - (n-1)\epsilon_n^m\chi_{n-1}^m + (n+2)\epsilon_{n+1}^m\chi_{n+1}^m \end{cases}$$

⋮

※ Triangle rule



Given a pair, find the other pair starting from $A(\lambda,\theta)$, $B(\lambda,\theta)$ to $C(\lambda,\theta)$, $D(\lambda,\theta)$ <grid to grid>

→ But all done using spectral method

$$A = \sum_m\sum_n A_n^m Y_n^m \quad \text{where } A = A(\lambda,\mu)$$

$$A_n^m = \frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} A Y_n^{m*}\, d\mu\, d\lambda$$

Two ways of truncation.

(1) Rhomboidal truncation

$$A(\lambda,\mu) = \sum_{m=-N}^{N}\sum_{n=|m|}^{|m|+N} A_n^m Y_n^m(\lambda,\mu)$$

(2) triangular truncation

$$A(\lambda,\mu) = \sum_{m=-N}^{N}\sum_{n=|m|}^{N} A_n^m Y_n^m(\lambda,\mu)$$

• Transformation from grid to spectral space

Step1 Perform the Fourier transform along latitudes

$$A^m(\mu) = \frac{1}{2\pi}\int_0^{2\pi} A(\lambda,\mu)e^{-im\lambda}\,d\lambda$$

Step2 Perform the Legendre transform.

$$A_n^m = \int_{-1}^{1} A^m(\mu)P_n^m(\mu)\,d\mu$$

• Transformation from spectral to grid space

Step1 Perform the reverse Legendre transform

$$A^m(\mu) = \sum_n A_n^m P_n^m(\mu_k)$$

Step2 Perform the reverse Fourier transform

$$A(\lambda_j,\mu_k) = \sum_m A^m(\mu_k)e^{im\lambda_j}$$

discrete form

$$A^m(\mu_k) = \frac{1}{2M}\sum_{j=0}^{2M-1} A(\lambda_j,\mu_k)e^{-im\lambda_j}$$

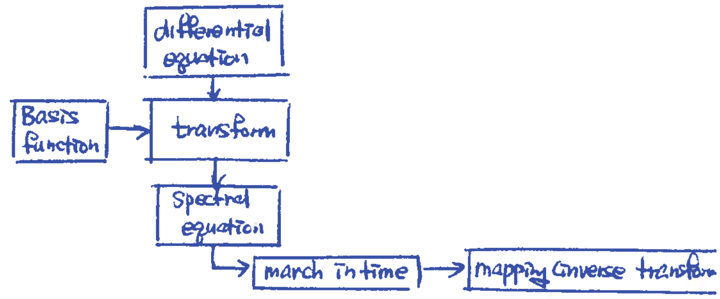
$$A_n^m = \sum_{k=1}^{K} W(\mu_k)A^m(\mu_k)P_n^m(\mu_k)$$

Gaussian weight

※ the use of Gaussian latitudes and weights enables one to calculate the Legendre transform exactly.

• Spectral transform method

given $\psi(\lambda,\mu,t=0)$ ⟶ solve $\psi(\lambda,\mu,t)$



$$\begin{cases} Y_n^m = P_n^m e^{im\lambda} \\ Y_n^{m*} = P_n^m e^{-im\lambda} \end{cases}$$

Grid to spectral ⟶ forward transform (Fourier, Legendre)

Spectral to grid ⟶ inverse transform (Legendre, Fourier)

The rules of spectral modeling

"philosophy of the spectral transform method"

i) all derivatives are carried out exactly

ii) all multiplications are done on the transform grid (grid space)

★ Barotropic Spectral model on a sphere — ★ EXAM

barotropic vorticity eq.

$$\frac{\partial}{\partial t}(\underbrace{\nabla^2\psi}_{\zeta}+f) = -J(\psi, \underbrace{\nabla^2\psi}_{\zeta}+f)$$

→ on a sphere.

$$\frac{\partial\zeta}{\partial t} = \frac{1}{a^2\cos\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\lambda}(\zeta+f) - \frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\theta}(\zeta+f)\right)$$

$$= \frac{1}{a^2}\left(\frac{\partial\psi}{\partial\mu}\frac{\partial}{\partial\lambda}(\zeta+f) - \frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\mu}(\zeta+f)\right) \qquad \because \mu = \sin\theta$$

↳ barotropic nonlinear vorticity eq. on a sphere.

$\because \frac{\partial f}{\partial\lambda} = 0$, $\frac{\partial f}{\partial\mu} = 2\Omega$ — there is no latitude dependence

$$\frac{\partial\zeta}{\partial t} = \frac{1}{a^2}\left(\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu}\right) - \boxed{\frac{2\Omega}{a^2}}\frac{\partial\psi}{\partial\lambda}$$

$$\frac{\partial\zeta}{\partial t} = F(\lambda,\mu) - \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} \qquad \text{①}$$

$$F(\lambda,\mu) = \frac{1}{a^2}\left(\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu}\right) \text{ : non-linear term}$$

Let's write

$$\begin{cases} \zeta(\lambda,\mu,t) = \sum_m\sum_n \zeta_n^m(t)Y_n^m(\lambda,\mu) \\ \psi(\lambda,\mu,t) = \sum_m\sum_n \psi_n^m(t)Y_n^m(\lambda,\mu) \\ F(\lambda,\mu,t) = \sum_m\sum_n F_n^m(t)Y_n^m(\lambda,\mu) \end{cases} \xrightarrow{\text{subst.}} \text{①}$$

$$\sum_m\sum_n \frac{d\zeta_n^m(t)}{dt}Y_n^m(\lambda,\mu) = \sum_m\sum_n\left(F_n^m(t)Y_n^m(\lambda,\mu) - \frac{2\Omega}{a^2}\psi_n^m(t)\frac{\partial}{\partial\lambda}Y_n^m(\lambda,\mu)\right)$$

using, $\zeta = \nabla^2\psi$

$$\therefore \frac{d\zeta_n^m(t)}{dt}Y_n^m(\lambda,\mu) = -\frac{n(n+1)}{a^2}\frac{d\psi_n^m(t)}{dt}Y_n^m(\lambda,t)$$

$$\frac{\partial}{\partial\lambda}Y_n^m(\lambda,\mu) = imY_n^m(\lambda,\mu)$$

$$\therefore \frac{d\psi_n^m}{dt} = \frac{2\Omega im}{n(n+1)}\psi_n^m - \frac{a^2}{n(n+1)}F_n^m$$

↳ spectral form of the barotropic vorticity eq.

First, Let's consider the non-linear term F_n^m

→ $\frac{\partial\psi}{\partial\lambda}$, $\frac{\partial\psi}{\partial\mu}$, $\frac{\partial\zeta}{\partial\lambda}$ & $\frac{\partial\zeta}{\partial\mu}$ are calculated on grid points by projecting the spectral coefficients onto the space domain. There are then multiplied to get values of the nonlinear terms $F(\lambda,\mu)$ on the grid points. Then perform the Fourier-Legendre transform of $F(\lambda,\mu)$ to get F_n^m

$$F(\lambda,\mu)=\frac{1}{a^2}\left(\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda}-\frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu}\right)$$
$$=\frac{1}{a^2(1-\mu^2)}\left((1-\mu^2)\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda}-\frac{\partial\psi}{\partial\lambda}(1-\mu^2)\frac{\partial\zeta}{\partial\mu}\right)$$

Let $\psi(\lambda,\mu)=\sum_m\sum_n\psi_n^m(t)Y_n^m(\lambda,\mu)$

so that

$$(1-\mu^2)\frac{\partial\psi}{\partial\mu}=\sum\sum\psi_n^m(t)e^{im\lambda}\boxed{(1-\mu^2)\frac{\partial}{\partial\mu}P_n^m(\mu)}$$

∴ recurrence relation

$$(1-\mu^2)\frac{d}{d\mu}P_n^m(\mu)=-n\epsilon_{n+1}^mP_{n+1}^m(\mu)+(n+1)\epsilon_n^mP_{n-1}^m(\mu)$$
$$\epsilon_n^m=\left(\frac{n^2-m^2}{4n^2-1}\right)^{1/2}$$

✓ $$(1-\mu^2)\frac{\partial\psi}{\partial\mu}=\underbrace{\sum_m\sum_n}_{\text{inverse transform}}\psi_n^m\left[-n\epsilon_{n+1}^mP_{n+1}^m(\mu)+(n+1)\epsilon_n^mP_{n-1}^m(\mu)\right]e^{im\lambda}$$

also

✓ $$\frac{\partial\psi}{\partial\lambda}=\sum_m\sum_n im\psi_n^m(t)P_n^m(\mu)e^{im\lambda}$$

Similarly for $(1-\mu^2)\frac{\partial\zeta}{\partial\mu}$ + $\frac{\partial\zeta}{\partial\lambda}$

★ Steps for integrating the barotropic vorticity eq

Step1 From the coefficients ψ_n^m and ζ_n^m, obtain the grid-point values of $(1-\mu^2)\frac{\partial\psi}{\partial\mu}$ and $(1-\mu^2)\frac{\partial\zeta}{\partial\mu}$ along a latitude circle

Step2 Similarly, obtain grid-point values of $\frac{\partial\psi}{\partial\lambda}$ and $\frac{\partial\zeta}{\partial\lambda}$ along the latitude circle.

Step3 Multiply $(1-\mu^2)\frac{\partial\psi}{\partial\mu}$, $\frac{\partial\zeta}{\partial\mu}$, $\frac{\partial\psi}{\partial\lambda}$, and $(1-\mu^2)\frac{\partial\zeta}{\partial\lambda}$ to compute the nonlinear term $F(\lambda,\mu)$ on the grid points (λ,μ)

Step4 Perform the Fourier transform of $F(\lambda,\mu)$ along the latitude circle to obtain the $F^m(\mu)$. This is done by using the FFT.

Step5 Perform the Legendre transform of $F^m(\mu)$ at the various latitudes to obtain F_n^m. This is done by using Gaussian quadrature.

Step6 From ψ_n^m and F_n^m, calculate the R.H.S. of the spectral vorticity equation and obtain $d\psi_n^m/dt$

Step7 From $\psi_n^m(t-\Delta t)$ and $d\psi_n^m/dt$, obtain $\psi_n^m(t+\Delta t)$ at the next time step as

$$\psi_n^m(t+\Delta t)=\psi_n^m(t-\Delta t)+2\Delta t\frac{d\psi_n^m}{dt}$$

- Compute $\frac{\partial A}{\partial\mu}$ spectrally.

given $A(\lambda,\mu)$, show all steps and a flow chart

$A(\lambda,\mu)$ in grid point

i) Perform the Fourier transform along latitudes
$$A^m(\mu)=\frac{1}{2\pi}\int_0^{2\pi}A(\lambda,\mu)e^{-im\lambda}d\lambda$$
ii) Perform the Legendre transform of $A^m(\mu)$
$$A_n^m=\int_{-1}^{1}A^m(\mu)P_n^m(\mu)d\mu$$

A_n^m

<Box of ϵ_n^m>

m →, n ↓

$$\epsilon_n^m=\left(\frac{n^2-m^2}{4n^2-1}\right)^{1/2}$$

<Box of $P_n^m(\mu)$>

m →, n ↓

$$P_n^m(\mu)=\frac{(1-\mu)^{m/2}}{2^nn!}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n,\quad|\mu|\le1$$

where $\mu=\sin\theta$

We just need to know P_0^0, then we use the recurrence relation of P_n^m's to get the rest of P_n^m's

<Box of $\frac{dP_n^m(\mu)}{d\mu}$>

m →, n ↓

$$\frac{dP_n^m}{d\mu}=\frac{1}{1-\mu^2}\left[-n\epsilon_{n+1}^mP_{n+1}^m(\mu)+(n+1)\epsilon_n^mP_{n-1}^m(\mu)\right]$$

$$\frac{\partial A}{\partial\mu}=\underbrace{\sum_m\sum_n}_{\text{inverse transform}}A_n^me^{im\lambda}\frac{\partial}{\partial\mu}P_n^m(\mu)$$

- How to calculate Jacobian (non-linear)

$$F(\lambda,\mu)=\frac{1}{a^2}\left(\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda}-\frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu}\right)$$

Given $\psi(\lambda,\mu)$, how do I compute $F_n^m(t)$

$\zeta(\lambda,\mu)\longrightarrow\zeta_n^m$
$\psi(\lambda,\mu)\longrightarrow\psi_n^m$

↑ Fourier-Legendre transform (using orthogonality property)

compute $\frac{\partial\psi}{\partial\mu}$, $\frac{\partial\zeta}{\partial\lambda}$, $\frac{\partial\psi}{\partial\lambda}$, $\frac{\partial\zeta}{\partial\mu}$ → spectrally in grid points

→ multiply → $F(\lambda,\mu)\xrightarrow[\text{transform}]{\text{Fourier-Legendre}}F_n^m(t)$

- Given $U(\lambda,\theta)$, $V(\lambda,\theta)$, solve for $\zeta(\lambda,\theta)$, $D(\lambda,\theta)$

$\boxed{U(\lambda,\theta),\ V(\lambda,\theta)}$

↓ Fourier-Legendre transform

$\boxed{U_n^m,\ V_n^m}$

↓ → look below eqs. → matrix form

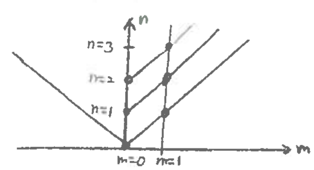
$\boxed{\zeta_n^m\quad D_n^m}$

↓ Inverse transform

$\boxed{\zeta(\lambda,\theta),\ D(\lambda,\theta)}$

$$n(n+1)U_n^m=-imD_n^m-(n+1)\epsilon_n^m\zeta_{n-1}^m+n\epsilon_{n+1}^m\zeta_{n+1}^m\quad—①$$
$$n(n+1)V_n^m=-im\zeta_n^m+(n+1)\epsilon_n^mD_{n-1}^m-n\epsilon_{n+1}^mD_{n+1}^m\quad—②$$
for $n\ge|m|$

Let's use the rhomboidal truncation, truncated at $n=3$



For $m=0$ i) $n=0$ U_0^0, V_0^0 → known

ii) $n=1$
$$\boxed{2U_1^0=-2\epsilon_1^0\zeta_0^0}+\epsilon_2^0\zeta_2^0$$
$$\boxed{2V_1^0=2\epsilon_1^0D_0^0}-\epsilon_2^0D_2^0$$
↳ known

Therefore, we can easily compute ζ_n^0, D_n^0 (for $m=0$)

Now, for $m=1$, eq ①,② can be written as

$$n(n+1)U_n^1=-iD_n^1-(n+1)\epsilon_n^1\zeta_{n-1}^1+n\epsilon_{n+1}^1\zeta_{n+1}^1$$
$$n(n+1)V_n^1=-i\zeta_n^1+(n+1)\epsilon_n^1D_{n-1}^1-n\epsilon_{n+1}^1D_{n+1}^1$$

i) $n=1$
$$2U_1^1=-iD_1^1-0+\epsilon_2^1\zeta_2^1$$
$$2V_1^1=-i\zeta_1^1+0-\epsilon_2^1D_2^1$$

ii) $n=2$
$$6U_2^1=-iD_2^1-3\epsilon_2^1\zeta_1^1+2\epsilon_3^1\zeta_3^1$$
$$6V_2^1=-i\zeta_2^1+3\epsilon_2^1\zeta_1^1-2\epsilon_3^1D_3^1$$

iii) $n=3$
$$12U_3^1=-iD_3^1-4\epsilon_3^1\zeta_2^1+\cancel{3\epsilon_4^1\zeta_4^1}$$
$$12V_3^1=-i\zeta_2^1+4\epsilon_3^1D_2^1-\cancel{3\epsilon_4^1D_4^1}$$
→ 0 (truncated at n=3)

Rearranging and writing it as a matrix form

$$\underbrace{\begin{pmatrix}U_1^1\\V_1^1\\U_2^1\\V_2^1\\U_3^1\\V_3^1\end{pmatrix}}_{\mathbf{U}}=\underbrace{\begin{pmatrix}0&-\frac{i}{2}&\frac{\epsilon_2^1}{2}&0&0&0\\-\frac{i}{2}&0&0&-\frac{\epsilon_2^1}{2}&0&0\\-\frac{\epsilon_2^1}{2}&0&0&-\frac{i}{6}&\frac{\epsilon_3^1}{3}&0\\0&\frac{\epsilon_2^1}{2}&-\frac{i}{6}&0&0&-\frac{\epsilon_3^1}{3}\\0&0&-\frac{\epsilon_3^1}{3}&0&0&-\frac{i}{12}\\0&0&0&\frac{\epsilon_3^1}{3}&-\frac{i}{12}&0\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}\zeta_1^1\\D_1^1\\\zeta_2^1\\D_2^1\\\zeta_3^1\\D_3^1\end{pmatrix}}_{Z}$$

$$\therefore\ Z=A^{-1}\mathbf{U}$$

★ Shallow-water spectral model

· look at p.7~8. for finite differences

· continuity eq.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \int_0^w dw=-\int_0^z\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz$$

$$\to w_z=-z\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

also $w_z=\frac{dz}{dt}=\frac{\partial z}{\partial t}+V_H\cdot\nabla z$

$$\to \frac{\partial z}{\partial t}=-V_H\cdot\nabla z-z\nabla\cdot V_H \quad \times g$$

$$\Rightarrow \frac{\partial \phi}{\partial t}=-V_H\cdot\nabla\phi-\phi\nabla\cdot V$$

$\because \phi=\bar{\Phi}+\phi'$ (time-invariant)

$$\frac{\partial\phi'}{\partial t}=-V_H\cdot\nabla(\bar{\Phi}+\phi')-(\bar{\Phi}+\phi')\nabla\cdot V_H$$
$$=-V_H\cdot\nabla\phi'-\phi'\nabla\cdot V_H-\bar{\Phi}\nabla\cdot V_H$$

or
$$\frac{\partial\phi'}{\partial t}=-\nabla\cdot(\phi' V_H)-\bar{\Phi}\nabla\cdot V_H \qquad ⓐ$$

· horiz. momentum eq.

$$\frac{\partial V_H}{\partial t}=-V_H\cdot\nabla V_H-f k\times V_H-\nabla\phi$$

$$\because V_H\cdot\nabla V_H=\nabla\left(\frac{V_H\cdot V_H}{2}\right)+\zeta k\times V_H$$

$$\frac{\partial V_H}{\partial t}=-(\zeta+f)k\times V_H-\nabla\left(\phi+\frac{V_H\cdot V_H}{2}\right)$$
$$=-(\zeta+f)k\times V_H-\nabla\left(\phi'+\frac{V_H\cdot V_H}{2}\right) \qquad ①$$

→ express this in terms of the vorticity and divergence (good for spectral)

→ vorticity eq. ($k\cdot\nabla\times①$)

$$\frac{\partial(k\cdot\nabla\times V_H)}{\partial t}=-k\cdot\nabla\times[(\zeta+f)k\times V_H]-k\cdot\nabla\times\nabla\left(\phi'+\frac{V_H\cdot V_H}{2}\right)$$

$$\Rightarrow \frac{\partial\zeta}{\partial t}=-\nabla\cdot(\zeta+f)V_H \qquad ⓑ$$

→ divergence eq. ($\nabla\cdot①$)

$$\frac{\partial(\nabla\cdot V_H)}{\partial t}=-\nabla\cdot[(\zeta+f)k\times V_H]-\nabla\cdot\nabla\left(\phi'+\frac{V_H\cdot V_H}{2}\right)$$

$$\to \frac{dD}{dt}=\frac{\partial U(\zeta+f)}{a\cos\theta\,\partial\lambda}-\frac{\partial U\cos\theta(\zeta+f)}{a\cos\theta\,\partial\theta}-\nabla^2\left(\phi'+\frac{V_H\cdot V_H}{2}\right) \qquad ⓒ$$

$\because U=\frac{u\cos\theta}{a},\quad V=\frac{v\cos\theta}{a}$: Robert function

ⓐ, ⓑ, ⓒ →

$$\begin{cases}\frac{\partial\zeta}{\partial t}=-\frac{1}{\cos^2\theta}\left(\frac{\partial U(\zeta+f)}{\partial\lambda}+\cos\theta\frac{\partial V(\zeta+f)}{\partial\theta}\right)\\ \frac{\partial D}{\partial t}=\frac{1}{\cos^2\theta}\left(\frac{\partial V(\zeta+f)}{\partial\lambda}-\cos\theta\frac{\partial U(\zeta+f)}{\partial\theta}\right)-\nabla^2\left(\frac{U^2+V^2}{2\cos^2\theta}+\phi'\right)\\ \frac{\partial\phi'}{\partial t}=-\frac{1}{\cos^2\theta}\left(\frac{\partial(U\phi')}{\partial\lambda}+\cos\theta\frac{\partial(V\phi')}{\partial\theta}\right)-\bar{\Phi}D\end{cases}$$

→

$$\begin{cases}\frac{\partial\nabla^2\psi}{\partial t}=-\frac{1}{\cos^2\theta}\left(\frac{\partial U(\nabla^2\psi+f)}{\partial\lambda}+\cos\theta\frac{\partial V(\nabla^2\psi+f)}{\partial\theta}\right)\\ \frac{\partial\nabla^2\chi}{\partial t}=\frac{1}{\cos^2\theta}\left(\frac{\partial V(\nabla^2\psi+f)}{\partial\lambda}-\cos\theta\frac{\partial U(\nabla^2\psi+f)}{\partial\theta}\right)-\nabla^2\left(\frac{U^2+V^2}{2\cos^2\theta}+\phi'\right)\\ \frac{\partial\phi'}{\partial t}=-\frac{1}{\cos^2\theta}\left(\frac{\partial(U\phi')}{\partial\lambda}+\cos\theta\frac{\partial(V\phi')}{\partial\theta}\right)-\bar{\Phi}D\end{cases}$$

Vorticity eq. does not produce gravity waves
Divergence eq. and cont. eq. produce gravity waves.

Expanding ψ, χ, ϕ', U and V spectrally →

$$\psi=a^2\sum_m\sum_n\psi_n^m Y_n^m$$

Similarly for χ, ϕ', U and V

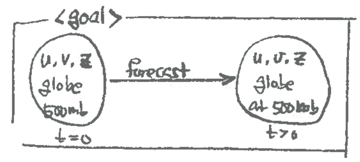
⇒ the spectral form of the shallow-water eqs

$$\begin{cases}-n(n+1)\frac{\partial\psi_n^m}{\partial t}=-\alpha_n^m(A,B)\\ -n(n+1)\frac{\partial\chi_n^m}{\partial t}=\alpha_n^m(B,-A)-\nabla^2\left(\frac{U^2+V^2}{2\cos^2\theta}+\phi'\right)_n^m\\ \frac{\partial\phi'^m_n}{\partial t}=-\alpha_n^m(U\phi',V\phi')-\bar{\Phi}D_n^m\end{cases}$$

where $A=U(\nabla^2\psi+f)$, $B=V(\nabla^2\psi+f)$, $D=\nabla\cdot V$, $V=Ui+Vj$

$$\alpha(A,B)=\frac{1}{\cos^2\theta}\left(\frac{\partial A}{\partial\lambda}+\cos\theta\frac{\partial B}{\partial\theta}\right)$$

· zonal u, meridional v, mass conti ϕ' → ψ, χ, ϕ'



· flow chart

given $u, v, z(\lambda,\theta,t=0)$ grid
↓
U, V, ϕ' $\because gz-gH$ grid
↓
U_n^m, V_n^m, ϕ'^m_n spectral space
↓ Δ problem
ζ_n^m, D_n^m ← $\psi_n^m, \chi_n^m, \phi'^m_n$ spectral space
↓
⋮

*difficult part

$$\alpha(A,B)=\frac{1}{\cos^2\theta}\left(\frac{\partial A}{\partial\lambda}+\cos\theta\frac{\partial B}{\partial\theta}\right)$$
$$=\frac{1}{\cos^2\theta}\left(\frac{\partial}{\partial\lambda}U(\nabla^2\psi+f)+\cos\theta\frac{\partial}{\partial\theta}V(\nabla^2\psi+f)\right)$$

$\frac{1}{\cos^2\theta}\frac{\partial}{\partial\lambda}\{U(\nabla^2\psi+f)\}$ start from grid space

ψ_n^m, χ_n^m spectral space
↓
$\nabla^2\psi_n^m, \nabla^2\chi_n^m$ spectral space
↓
$\nabla^2\psi, \nabla^2\chi$ grid

$C\equiv\frac{1}{\cos^2\theta}U(\nabla^2\psi+f)$: grid space
↓ Fourier-Legendre transform
C_n^m

*spectral rules library

given $Z(\lambda,\theta)\xrightarrow{\text{FFT}}Z^m(\theta)\xrightarrow{\text{Legendre transform}}Z_n^m$

$\frac{\partial}{\partial\lambda}$, $\frac{\partial}{\partial\theta}$ } spectrally

always multiply $A\times B$ on a grid.

To go from Z_n^m to $Z(\lambda,\theta)$: Inverse transform (mapping)
$\sum\sum Z_n^m\longrightarrow Z(\lambda,\theta)$

- nonlinear term treatment. multiplication must be computed in grid space; $\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\theta}$ must be treated spectrally.

$$-\frac{1}{\cos^2\theta}\frac{\partial}{\partial\lambda}U(\nabla^2\psi+f) = -\frac{\partial}{\partial\lambda}\left\{\frac{U(\nabla^2\psi+f)}{\cos^2\theta}\right\}$$

Start from $u(\lambda,\theta)$, $v(\lambda,\theta)$ grid space

↓

$U(\lambda,\theta)$, $V(\lambda,\theta)$ using $U=\frac{u\cos\theta}{a}$; $V=\frac{v\cos\theta}{a}$

↓

U_n^m V_n^m Fourier-Legendre transform

↓

ψ_n^m χ_n^m Δ problem

↓

$\nabla^2\psi_n^m$ $\nabla^2\chi_n^m$

↓ use inverse transform and go back to grid space

$\nabla^2\psi$, $\nabla^2\chi$

↓

$\nabla^2\psi+f$ on grid

$\frac{U(\nabla^2\psi+f)}{\cos^2\theta}$ on grid

↓ do Fourier-Legendre transform

$\left\{\frac{U(\nabla^2\psi+f)}{\cos^2\theta}\right\}_n^m$

↓

$-\frac{\partial}{\partial\lambda}\left\{\frac{U(\nabla^2\psi+f)}{\cos^2\theta}\right\}_n^m$ spectrally.

- Semi-implicit shallow-water spectral model

where are we going?

Three eqs u, v, ϕ } shallow water eq

next write semi-implicit form

eliminate and obtain one eq., one unknowns

That is a Helmholtz eq for D. you solve that for D

They then readily gives you u, v, ϕ'

The linear terms are integrated implicitly, while the nonlinear terms are integrated explicitly.

vorticity eq → explicitly

divergence + conti → implicitly

divergence eq → $\frac{\partial}{\partial t}D = F_1(\psi) - \nabla^2\phi'$ ——①

continuity eq → $\frac{\partial}{\partial t}\phi' = F_2(\phi') - \Phi D$ ——②

where $F_1(\psi) = \frac{1}{\cos^2\theta}\left(\frac{\partial}{\partial\lambda}V(\nabla^2\psi+f) - \cos\theta\frac{\partial}{\partial\theta}U(\nabla^2\psi+f)\right) - \nabla^2\left(\frac{U^2+V^2}{2\cos^2\theta}\right)$

$F_2(\phi') = -\frac{1}{\cos^2\theta}\left(\frac{\partial}{\partial\lambda}(U\phi') + \cos\theta\frac{\partial}{\partial\theta}(V\phi')\right)$

Let's define a time-average operator $\overline{(\ )}^t$ as

$$\overline{D}^t = \frac{D^{t+\Delta t}+D^{t-\Delta t}}{2}$$

①→ $\frac{D^{t+\Delta t}-D^{t-\Delta t}}{2\Delta t} = \frac{\overline{D}^t - D^{t-\Delta t}}{\Delta t} = F_1^t(\psi) - \nabla^2\overline{\phi}^t$

②→ $\frac{\phi^{t+\Delta t}-\phi^{t-\Delta t}}{2\Delta t} = \frac{\overline{\phi}^t - \phi^{t-\Delta t}}{\Delta t} = F_2^t(\phi') - \Phi\overline{D}^t$

⇒ $\overline{D}^t = D^{t-\Delta t} + F_1^t(\psi)\Delta t - \nabla^2\overline{\phi}^t\Delta t$ ——①'

$\overline{\phi}^t = \phi^{t-\Delta t} + F_2^t(\phi')\Delta t - \Phi\overline{D}^t\Delta t$ ——②'

eliminate $\overline{\phi}^t$ ($\nabla^2②/\Delta t \to ①'$)

$$\overline{D}^t = D^{t-\Delta t} + F_1^t(\psi)\Delta t - \Delta t\nabla^2\phi'^{t-\Delta t} - (\Delta t)^2\nabla^2F_2^t(\phi') + (\Delta t)^2\Phi\nabla^2\overline{D}^t$$

rearranging,

$$(\Delta t)^2\Phi\nabla^2\overline{D}^t - \overline{D}^t = F_3(D,\phi',\psi)^{t,t-\Delta t} \quad ——⊛$$

→ Helmholtz eq.

where

$$F_3(D,\phi',\psi)^{t,t-\Delta t} = (\Delta t)^2\nabla^2F_2^t(\phi') - \Delta t F_1^t(\psi) + (\Delta t)\nabla^2\phi^{t-\Delta t} - D^{t-\Delta t}$$

We can write ⊛ in spectral form as

$$-(\Delta t)^2\Phi\frac{n(n+1)}{a^2}\overline{D}_n^{m\,t} - \overline{D}_n^{m\,t} = F_{3n}^m$$

or

$$\overline{D}_n^{m\,t} = -\frac{F_{3n}^m}{1+\left(\frac{n(n+1)}{a^2}(\Delta t)^2\Phi\right)} \quad ——③$$

∴ $D_n^{m\,t+\Delta t} = 2\overline{D}_n^{m\,t} - D_n^{m\,t-\Delta t}$

Substituting ③ into ②' → we can obtain $\overline{\Phi}_n^{m\,t}$ → hence $\phi_n^{m\,t+\Delta t}$

The $\zeta_n^{m\,t+\Delta t}$ can be obtained explicitly from the spectral form of the vorticity eq.

how to solve spectrally?

- Given $u(\lambda,\theta)$, $v(\lambda,\theta)$, $z(\lambda,\theta)$ at $t=0$
↓ Step? (flow chart)
End at $u(\lambda,\theta)$, $v(\lambda,\theta)$, $z(\lambda,\theta)$ at $t=\Delta t$

using shallow-water model via semi-implicit scheme.

given $u(\lambda,\theta)$, $v(\lambda,\theta)$, $z(\lambda,\theta)$ at $t=0$

↓

$U(\lambda,\theta)$, $V(\lambda,\theta)$, $\phi'(\lambda,\theta)$ at $t=0$

where $U=\frac{u\cos\theta}{a}$, $V=\frac{v\cos\theta}{a}$, $\phi'=gz-gH$ (gH → Φ)

↓ Fourier-Legendre transform

U_n^m V_n^m $\phi_n'^m$ at $t=0$

↓

ψ_n^m, χ_n^m, $\phi_n'^m$; ζ_n^m, D_n^m at $t=0$

where $\zeta_n^m = \nabla^2\psi_n^m = -n(n+1)\psi_n^m$, $D_n^m = \nabla^2\chi_n^m = -n(n+1)\chi_n^m$

nonlinear term

$\overline{D}_n^{m\,t} = -\frac{F_{3n}^m}{1+\left(\frac{n(n+1)}{a^2}(\Delta t)^2\Phi\right)}$

$D_n^{m\,t+\Delta t} = 2\overline{D}_n^{m\,t} - D_n^{m\,t-\Delta t}$

nonlinear term

(vorticity eq)

$-n(n+1)\frac{\partial\psi_n^m}{\partial t} = -\alpha_n^m(A,B)$

explicitly

$\psi_n^{m\,t+\Delta t} = \psi_n^{m(t-\Delta t)} + F$

nonlinear term

$\overline{\phi}_n^{m\,t} = \phi_n^{m\,t-\Delta t} + F_{2n}^{m\,t}\Delta t - \Phi\overline{D}_n^{m\,t}\Delta t$

$\phi_n'^{m\,t+\Delta t} = 2\overline{\phi}_n^{m\,t} - \phi_n'^{m\,t-\Delta t}$

↓

D_n^m, ζ_n^m, $\phi_n'^m$

↓ Δ problem

ψ_n^m, χ_n^m, $\phi_n'^m$ at $t=\Delta t$

↓ Δ problem

U_n^m V_n^m $\phi_n'^m$ at $t=\Delta t$

↓ Inverse transform

$U(\lambda,\theta)$, $V(\lambda,\theta)$, $\phi'(\lambda,\theta)$ at $t=\Delta t$

$U=\frac{u\cos\theta}{a}$, $V=\frac{v\cos\theta}{a}$, $\phi'=gz-gH$

↓

$u(\lambda,\theta)$, $v(\lambda,\theta)$, $z(\lambda,\theta)$ at $t=\Delta t$

Multilevel Global spectral model

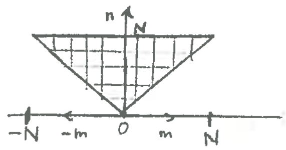
- Truncation in a spectral model

$$\zeta(\lambda,\mu) = \sum_m \sum_n \zeta_n^m Y_n^m(\lambda,\mu)$$

In general, $-\infty \le m \le \infty$, $|m| \le n \le \infty$

- triangular truncation ((n−m) varies) — # of zeros

$$\zeta(\lambda,\mu) = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \zeta_n^m Y_n^m(\lambda,\mu)$$



$$\int_{-1}^{1} P_{n_1}^{m_1} P_{n_2}^{m_2} P_{n_3}^{m_3} d\mu$$

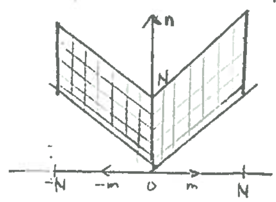
degree $N + N + N$

$\therefore$ # of points to use $= \frac{3N+1}{2}$

- Rhomboidal truncation (n−m fixed)

$$\zeta(\lambda,\mu) = \sum_{m=-N}^{N} \sum_{n=|m|}^{|m|+J} \zeta_n^m Y_n^m$$

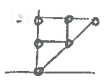
If $J = N$, the truncation is called rhomboidal
$J \ne N$, " " " " parallelogramic truncation



# of points to use $= \frac{2N+3J+1}{2}$

if $J=N$, then $\frac{5N+1}{2}$

- Total number of P_n^m for a triangular truncation



Just one side

$$\frac{(N+1)(N+2)}{2}$$

$m=0 \rightarrow N+1$
$m=1 \rightarrow N$
$\vdots$
$m=N \rightarrow 1$

$\rightarrow$ on average $\frac{N+2}{2}$ per zonal wavenumber — total zonal wavenumbers $= (N+1)$

EX) T-42 $\rightarrow \frac{43 \times 44}{2} = 946$ spectral components

- Total number of P_n^m for a rhomboidal truncation

$$(N+1)(N+1)$$

EX) R-42 $\rightarrow 43 \times 43 = 1849$ P_n^m

- Transform method

$A(\lambda,\mu) B(\lambda,\mu)$ nonlinear term

1. Perform a spectral to grid point transform of the model variables

(a) $A^m(\mu) = \sum_{n=|m|}^{N} A_n^m P_n^m(\mu)$
$B^m(\mu) = \sum_{n=|m|}^{N} B_n^m P_n^m(\mu)$: Inverse Legendre transform

(b) $A(\lambda,\mu) = \sum_{m=-M}^{M} A^m(\mu) e^{im\lambda}$
$B(\lambda,\mu) = \sum_{m=-M}^{M} B^m(\mu) e^{im\lambda}$: Inverse Fourier transform

2. Perform a calculation of the nonlinear products on grid points

$$C(\lambda,\mu) = A(\lambda,\mu) B(\lambda,\mu)$$

3. Perform a spectral transform of the nonlinear products at grid points.

$\Rightarrow$ alias-free calculations

* the minimum # of Gaussian latitudes for Legendre transform

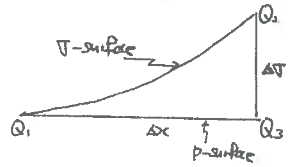
Rhomboidal $\rightarrow k = \frac{5N+1}{2}$

triangular $\rightarrow k = \frac{3N+1}{2}$

- The x-y-σ Coordinate system

Surface topography

$\sigma = \frac{P}{P_s}$ $\quad \sigma = \begin{cases} 1 & \text{at the earth's surface} \\ 0 & \text{at TOA} \end{cases}$



$$\frac{Q_3-Q_1}{\Delta x} = \frac{Q_2-Q_1}{\Delta x} + \frac{Q_3-Q_2}{\Delta\sigma}\frac{\Delta\sigma}{\Delta x}$$

$\because \Delta x \rightarrow 0$, $\Delta\sigma \rightarrow 0$

$$\left.\frac{\partial Q}{\partial x}\right|_p = \left.\frac{\partial Q}{\partial x}\right|_\sigma + \frac{\partial Q}{\partial \sigma}\left.\frac{\partial \sigma}{\partial x}\right|_p \quad \text{——— ①}$$

Similarly

$$\left.\frac{\partial Q}{\partial y}\right|_p = \left.\frac{\partial Q}{\partial y}\right|_\sigma + \frac{\partial Q}{\partial \sigma}\left.\frac{\partial \sigma}{\partial y}\right|_p \quad \text{——— ②}$$

$$\left.\frac{\partial Q}{\partial t}\right|_p = \left.\frac{\partial Q}{\partial t}\right|_\sigma + \frac{\partial Q}{\partial \sigma}\left.\frac{\partial \sigma}{\partial t}\right|_p \quad \text{——— ③}$$

$$\frac{\partial Q}{\partial p} = \frac{\partial Q}{\partial \sigma}\frac{\partial \sigma}{\partial p} \quad \text{——— ④}$$

- Substantial time derivative (d/dt) in σ-coord

x, y, p system

$$\frac{dQ}{dt} = \left.\frac{\partial Q}{\partial t}\right|_p + u\left.\frac{\partial Q}{\partial x}\right|_p + v\left.\frac{\partial Q}{\partial y}\right|_p + \omega\frac{\partial Q}{\partial p}$$

$$\frac{dQ}{dt} = \left.\frac{\partial Q}{\partial t}\right|_\sigma + u\left.\frac{\partial Q}{\partial x}\right|_\sigma + v\left.\frac{\partial Q}{\partial y}\right|_\sigma + \left(\left.\frac{\partial \sigma}{\partial t}\right|_p + u\left.\frac{\partial \sigma}{\partial x}\right|_p + v\left.\frac{\partial \sigma}{\partial y}\right|_p + \omega\frac{\partial \sigma}{\partial p}\right)\frac{\partial Q}{\partial \sigma}$$

$$= \left.\frac{\partial Q}{\partial t}\right|_\sigma + u\left.\frac{\partial Q}{\partial x}\right|_\sigma + v\left.\frac{\partial Q}{\partial y}\right|_\sigma + \frac{\partial Q}{\partial \sigma}\frac{d\sigma}{dt}$$

$$\therefore \frac{dQ}{dt} = \left.\frac{\partial Q}{\partial t}\right|_\sigma + u\left.\frac{\partial Q}{\partial x}\right|_\sigma + v\left.\frac{\partial Q}{\partial y}\right|_\sigma + \dot\sigma\frac{\partial Q}{\partial \sigma} \quad \because \dot\sigma = \frac{d\sigma}{dt} \quad \text{——— ⑤}$$

- Pressure gradient term in σ-coord.

$$-g\left.\frac{\partial z}{\partial x}\right|_p = -g\left.\frac{\partial z}{\partial x}\right|_\sigma - g\left.\frac{\partial \sigma}{\partial x}\right|_p\frac{\partial z}{\partial \sigma}$$

$\because$ using hydrostatic eq $\left(\frac{\partial p}{\partial z} = -\rho g\right)$ $(p = \rho RT)$

$$g\frac{\partial z}{\partial \sigma} = g\frac{\partial z}{\partial p}\frac{\partial p}{\partial \sigma} = -\frac{RT}{p}\frac{\partial p}{\partial \sigma}$$

Hence

$$-g\left.\frac{\partial z}{\partial x}\right|_p = -g\left.\frac{\partial z}{\partial x}\right|_\sigma + \frac{RT}{p}\left.\frac{\partial \sigma}{\partial x}\right|_p\frac{\partial p}{\partial \sigma}$$

$\because$ Let $Q = p$ in eq ① and noting $\left.\frac{\partial p}{\partial x}\right|_p = 0$,

$$0 = \left.\frac{\partial p}{\partial x}\right|_\sigma + \frac{\partial p}{\partial \sigma}\left.\frac{\partial \sigma}{\partial x}\right|_p$$

$$\therefore -g\left.\frac{\partial z}{\partial x}\right|_p = -g\left.\frac{\partial z}{\partial x}\right|_\sigma - \frac{RT}{p}\left.\frac{\partial p}{\partial x}\right|_\sigma \quad \text{——— ⑥}$$

Similarly

$$-g\left.\frac{\partial z}{\partial y}\right|_p = -g\left.\frac{\partial z}{\partial y}\right|_\sigma - \frac{RT}{p}\left.\frac{\partial p}{\partial y}\right|_\sigma \quad \text{——— ⑦}$$

(two terms large! difference small!) $\hookrightarrow$ numerical expression important

Here, let's introduce the Exner function

$$\Pi = \frac{T}{\theta} = \left(\frac{p}{p_0}\right)^{R/C_p} \quad \text{——— ⑧}$$

$$\frac{\partial}{\partial x}\ln ⑧ \rightarrow \frac{1}{\pi}\left.\frac{\partial \Pi}{\partial x}\right|_\sigma = \frac{R}{PC_p}\left.\frac{\partial P}{\partial x}\right|_\sigma$$

⑥ $\rightarrow$

$$-g\left.\frac{\partial z}{\partial x}\right|_p = -g\left.\frac{\partial z}{\partial x}\right|_\sigma - C_p\frac{T}{\Pi}\left.\frac{\partial \Pi}{\partial x}\right|_\sigma$$

$(\because T = \theta\Pi)$

$$\therefore -g\left.\frac{\partial z}{\partial x}\right|_p = -g\left.\frac{\partial z}{\partial x}\right|_\sigma - C_p\theta\left.\frac{\partial \Pi}{\partial x}\right|_\sigma \quad \text{——— ⑨}$$

Similarly

$$-g\left.\frac{\partial z}{\partial y}\right|_p = -g\left.\frac{\partial z}{\partial y}\right|_\sigma - C_p\theta\left.\frac{\partial \Pi}{\partial y}\right|_\sigma \quad \text{——— ⑩}$$

• Hydrostatic eq. in σ-coord.

$$\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{1}{\rho} = -\frac{RT}{p}$$

$$\frac{\partial \Phi}{\partial \sigma}\frac{\partial \sigma}{\partial p} = -\frac{RT}{p} \qquad \because \frac{\partial \sigma}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p}{p_s}\right) = \frac{1}{p_s}$$

$$\frac{p}{p_s}\frac{\partial \Phi}{\partial \sigma} = -RT$$

or

$$\sigma \frac{\partial \Phi}{\partial \sigma} = -RT \qquad \text{—— (11)}$$

An alternative form

$$\frac{\partial}{\partial \sigma}\ln\theta \longrightarrow \frac{1}{\pi}\frac{\partial \pi}{\partial \sigma} = \frac{R}{c_p p}\frac{\partial p}{\partial \sigma}$$

$$\because \frac{\partial p}{\partial z} = -g\rho = -g\left(\frac{p}{RT}\right)$$

$$\frac{1}{\pi}\frac{\partial \pi}{\partial \sigma} = -\frac{g}{c_p T}\frac{\partial z}{\partial \sigma} \qquad \because T = \theta\pi$$

or

$$g\frac{\partial z}{\partial \sigma} = -c_p\theta\frac{\partial \pi}{\partial \sigma} \qquad \text{—— (13)}$$

★ mass continuity eq. in σ-coord

x-y-p coord.

$$\left.\frac{\partial u}{\partial x}\right|_p + \left.\frac{\partial v}{\partial y}\right|_p + \frac{\partial \omega}{\partial p} = 0$$

$\because$ (1) → $\left.\frac{\partial u}{\partial x}\right|_p = \left.\frac{\partial u}{\partial x}\right|_\sigma + \frac{\partial u}{\partial \sigma}\left.\frac{\partial \sigma}{\partial x}\right|_p$ ✓

(2) → $\left.\frac{\partial v}{\partial y}\right|_p = \left.\frac{\partial v}{\partial y}\right|_\sigma + \frac{\partial v}{\partial \sigma}\left.\frac{\partial \sigma}{\partial y}\right|_p$ ✓

$$\frac{\partial \omega}{\partial p} = \frac{\partial}{\partial p}\left(\frac{dp}{dt}\right) = \frac{\partial}{\partial \sigma}\left(\frac{dp}{dt}\right)\frac{\partial \sigma}{\partial p}$$

$$= \frac{\partial}{\partial \sigma}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \dot\sigma\frac{\partial p}{\partial \sigma}\right)\frac{\partial \sigma}{\partial p}$$

$$= \left[\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \dot\sigma\frac{\partial}{\partial \sigma}\right)\frac{\partial p}{\partial \sigma} + \frac{\partial u}{\partial \sigma}\left.\frac{\partial p}{\partial x}\right|_\sigma + \frac{\partial v}{\partial \sigma}\left.\frac{\partial p}{\partial y}\right|_\sigma + \frac{\partial \dot\sigma}{\partial \sigma}\frac{\partial p}{\partial \sigma}\right]\frac{\partial \sigma}{\partial p}$$

or

$$\frac{\partial \omega}{\partial p} = \frac{\partial \sigma}{\partial p}\frac{d}{dt}\left(\frac{\partial p}{\partial \sigma}\right) + \left(\frac{\partial u}{\partial \sigma}\left.\frac{\partial p}{\partial x}\right|_\sigma + \frac{\partial v}{\partial \sigma}\left.\frac{\partial p}{\partial y}\right|_\sigma + \frac{\partial \dot\sigma}{\partial \sigma}\frac{\partial p}{\partial \sigma}\right)\frac{\partial \sigma}{\partial p}$$

Letting $Q = p$ in (1) + (2).

$$\because \; 0 = \left.\frac{\partial p}{\partial x}\right|_p = \left.\frac{\partial p}{\partial x}\right|_\sigma + \frac{\partial p}{\partial \sigma}\left.\frac{\partial \sigma}{\partial x}\right|_p \longrightarrow \left.\frac{\partial p}{\partial x}\right|_\sigma = -\frac{\partial p}{\partial \sigma}\left.\frac{\partial \sigma}{\partial x}\right|_p$$

$$0 = \left.\frac{\partial p}{\partial y}\right|_p = \left.\frac{\partial p}{\partial y}\right|_\sigma + \frac{\partial p}{\partial \sigma}\left.\frac{\partial \sigma}{\partial y}\right|_p \longrightarrow \left.\frac{\partial p}{\partial y}\right|_\sigma = -\frac{\partial p}{\partial \sigma}\left.\frac{\partial \sigma}{\partial y}\right|_p$$

So $$\frac{\partial \omega}{\partial p} = \frac{\partial \sigma}{\partial p}\frac{d}{dt}\left(\frac{\partial p}{\partial \sigma}\right) - \frac{\partial p}{\partial \sigma}\left(\left.\frac{\partial \sigma}{\partial x}\right|_p\frac{\partial u}{\partial \sigma} + \left.\frac{\partial \sigma}{\partial y}\right|_p\frac{\partial v}{\partial \sigma} - \frac{\partial \dot\sigma}{\partial \sigma}\right)\frac{\partial \sigma}{\partial p}$$ ✓

$$\left.\frac{\partial u}{\partial x}\right|_\sigma + \cancel{\frac{\partial u}{\partial \sigma}\left.\frac{\partial \sigma}{\partial x}\right|_p} + \left.\frac{\partial v}{\partial y}\right|_\sigma + \cancel{\frac{\partial v}{\partial \sigma}\left.\frac{\partial \sigma}{\partial y}\right|_p} + \frac{\partial \sigma}{\partial p}\frac{d}{dt}\left(\frac{\partial p}{\partial \sigma}\right) - \frac{\partial p}{\partial \sigma}\left(\cancel{\left.\frac{\partial \sigma}{\partial x}\right|_p\frac{\partial u}{\partial \sigma}} + \cancel{\left.\frac{\partial \sigma}{\partial y}\right|_p\frac{\partial v}{\partial \sigma}} - \frac{\partial \dot\sigma}{\partial \sigma}\right)\frac{\partial \sigma}{\partial p} = 0$$

or

$$\left.\frac{\partial u}{\partial x}\right|_\sigma + \left.\frac{\partial v}{\partial y}\right|_\sigma + \frac{\partial \dot\sigma}{\partial \sigma} + \frac{\partial \sigma}{\partial p}\frac{d}{dt}\left(\frac{\partial p}{\partial \sigma}\right) = 0$$

$$\because \left(\sigma = \frac{p}{p_s}, \; \frac{\partial p}{\partial \sigma} = p_s\right)$$

$$\frac{1}{p_s}\frac{dp_s}{dt} + \left.\frac{\partial u}{\partial x}\right|_\sigma + \left.\frac{\partial v}{\partial y}\right|_\sigma + \frac{\partial \dot\sigma}{\partial \sigma} = 0$$

$$\therefore \frac{d\ln p_s}{dt} + \left.\frac{\partial u}{\partial x}\right|_\sigma + \left.\frac{\partial v}{\partial y}\right|_\sigma + \frac{\partial \dot\sigma}{\partial \sigma} = 0$$

Flux form →

$$\frac{dp_s}{dt} + p_s\left(\left.\frac{\partial u}{\partial x}\right|_\sigma + \left.\frac{\partial v}{\partial y}\right|_\sigma + \frac{\partial \dot\sigma}{\partial \sigma}\right) = 0$$

or

$$\frac{\partial p_s}{\partial t} + u\left.\frac{\partial p_s}{\partial x}\right|_\sigma + v\left.\frac{\partial p_s}{\partial y}\right|_\sigma + \dot\sigma\frac{\partial p_s}{\partial \sigma} + p_s\left.\frac{\partial u}{\partial x}\right|_\sigma + p_s\left.\frac{\partial v}{\partial y}\right|_\sigma + p_s\frac{\partial \dot\sigma}{\partial \sigma} = 0$$

$$\therefore \frac{\partial p_s}{\partial t} + \frac{\partial}{\partial x}(up_s) + \frac{\partial}{\partial y}(vp_s) + \frac{\partial}{\partial \sigma}(\dot\sigma p_s) = 0$$

↳ prognostic eq. in σ-coord.

• A closed system of eqs in σ-coordinates on a sphere

Spherical coord. system (λ, θ, r), $r = a + z$

$$\delta x = r\cos\theta\,\delta\lambda, \quad \delta y = r\,\delta\theta, \quad \delta z = \delta r$$

$\because z \ll a$, $r \approx a$

$$\delta x = a\cos\theta\,\delta\lambda, \quad \delta y = a\,\delta\theta, \quad \delta z = \delta r$$

↳ $\delta z = -\frac{\delta p}{g\rho}$ (in pressure coord.)

The eqs for a global model in (λ, θ, p)

→ horizontal momentum eq.

$$\frac{\partial \mathbf{V}}{\partial t} = -(\mathbf{V}\cdot\nabla)\mathbf{V} - \omega\frac{\partial \mathbf{V}}{\partial p} - f\mathbf{k}\times\mathbf{V} - \nabla\phi + \mathbf{F} \qquad \text{(A)}$$

→ hydrostatic eq.

$$\frac{\partial p}{\partial \phi} = -\rho \longrightarrow \frac{\partial \phi}{\partial p} = -\frac{RT}{p}(1 + 0.61q) \qquad \text{(B)}$$

→ thermodynamic eq

$$\frac{\partial T}{\partial t} = -\mathbf{V}\cdot\nabla T - \omega\underbrace{\left(\frac{\partial T}{\partial p} - \frac{RT}{c_p p}\right)}_{\text{adiabatic term}} + \frac{\dot Q}{c_p} \qquad \text{(C)}$$

→ mass continuity eq.

$$\nabla\cdot\mathbf{V} + \frac{\partial \omega}{\partial p} = 0 \qquad \text{(D)}$$

→ eq. of state

$$p = \rho RT \qquad \text{(E)}$$

6 eqs, 6 unknowns

u, v, ω, ρ, T, z

Top and bottom B.C.s for $\dot\sigma$ → $\dot\sigma = 0$ at $\sigma = 0$ & $\sigma = 1$

From eq. (6) + (7) →

$$-\nabla_p\phi = -\nabla_\sigma\phi - \frac{RT}{p}\nabla_\sigma p = -\nabla_\sigma\phi - RT\nabla_\sigma\ln p$$

$$\because (p = \sigma p_s)$$

$$-\nabla_p\phi = -\nabla_\sigma\phi - RT\nabla_\sigma\ln p_s$$

Thus, momentum eq →

$$\frac{\partial \mathbf{V}}{\partial t} = -(\mathbf{V}\cdot\nabla)\mathbf{V} - \dot\sigma\frac{\partial \mathbf{V}}{\partial \sigma} - f\mathbf{k}\times\mathbf{V} - (\nabla\phi + RT\nabla\ln p_s) + \mathbf{F}$$

$$\because \left((\mathbf{V}\cdot\nabla)\mathbf{V} = \nabla\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) + (\nabla\times\mathbf{V})\times\mathbf{V} = \nabla\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) + \zeta\mathbf{k}\times\mathbf{V}\right)$$

✓ $$\therefore \frac{\partial \mathbf{V}}{\partial t} = -(\zeta + f)\mathbf{k}\times\mathbf{V} - \dot\sigma\frac{\partial \mathbf{V}}{\partial \sigma} - \nabla\left(\phi + \frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) - RT\nabla\ln p_s + \mathbf{F} \qquad \text{—— (I)}$$

hydrostatic eq →

✓ $$\sigma\frac{\partial \phi}{\partial \sigma} = -RT \qquad \text{—— (II)}$$

continuity eq →

✓ $$\frac{\partial \ln p_s}{\partial t} = -\mathbf{V}\cdot\nabla\ln p_s - \nabla\cdot\mathbf{V} - \frac{\partial \dot\sigma}{\partial \sigma} \qquad \left(\because \frac{\partial \ln p_s}{\partial \sigma} = 0\right) \qquad \text{—— (III)}$$

thermodynamic eq →

$$c_p\frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} = \dot Q$$

$$\because \frac{dp}{dt} = \frac{d(\sigma p_s)}{dt} = \dot\sigma p_s + \sigma\dot p_s$$

all diabatic heating

$$\frac{\partial T}{\partial t} = -\mathbf{V}\cdot\nabla T - \dot\sigma\frac{\partial T}{\partial \sigma} + \frac{RT}{c_p p}(\dot\sigma p_s + \sigma\dot p_s) + H_T = \frac{\dot Q}{c_p}$$

$$= -\mathbf{V}\cdot\nabla T - \dot\sigma\left(\frac{\partial T}{\partial \sigma} - \frac{RT}{c_p\sigma}\right) + \frac{RT}{c_p p_s}\frac{dp_s}{dt} + H_T$$

$$= -\mathbf{V}\cdot\nabla T - \dot\sigma\left(\frac{\partial T}{\partial \sigma} - \frac{RT}{c_p\sigma}\right) + \frac{RT}{c_p}\frac{d\ln p_s}{dt} + H_T$$

continuity

✓ $$\therefore \frac{\partial T}{\partial t} = -\mathbf{V}\cdot\nabla T + \dot\sigma\gamma - \frac{RT}{c_p}\left(\nabla\cdot\mathbf{V} + \frac{\partial \dot\sigma}{\partial \sigma}\right) + H_T \qquad \text{—— (IV)}$$

where $\gamma = \frac{RT}{c_p\sigma} - \frac{\partial T}{\partial \sigma}$

eq. of state →

✓ $$p = \rho RT \qquad \text{—— (V)}$$

∴ (I), (II), (III), (IV) and (V) represent a closed system of equations in $(\lambda, \theta, \sigma)$ coordinates

One more moisture eq. /

For a global spectral model, it is convenient to write the horizontal momentum eq. (I) as vorticity and divergence eqs (flux form)

$$\frac{\partial \zeta}{\partial t} = -\nabla\cdot(\zeta+f)\underline{V} - \mathbb{k}\cdot\nabla\times\left(RT\nabla q + \dot{\sigma}\frac{\partial \underline{V}}{\partial \sigma} - \underline{F}\right)$$

$$\frac{\partial D}{\partial t} = \mathbb{k}\cdot\nabla\times(\zeta+f)\underline{V} - \nabla\cdot\left(RT\nabla q + \dot{\sigma}\frac{\partial \underline{V}}{\partial \sigma} - \underline{F}\right) - \nabla^2\left(\phi + \frac{\underline{V}\cdot\underline{V}}{2}\right)$$

$$\frac{\partial T}{\partial t} = -\nabla\cdot(T\underline{V}) + TD + \dot{\sigma}\gamma - \frac{RT}{C_p}\left(D + \frac{\partial \dot{\sigma}}{\partial \sigma}\right) + H_T$$

$$\frac{\partial q}{\partial t} = -\underline{V}\cdot\nabla q - D - \frac{\partial \dot{\sigma}}{\partial \sigma}$$

$$\sigma\frac{\partial \phi}{\partial \sigma} = -RT$$

$$\frac{\partial r}{\partial t} = -\nabla\cdot(r\underline{V}) + rD - \dot{\sigma}\frac{\partial r}{\partial \sigma} + M$$ → moisture sinks and sources

where $q = \ln p_*$, r = specific humidity

Physical Processes

① The PBL

• **Bulk aerodynamic calculations over oceans and land**

the surface fluxes of sensible heat, water vapor, and momentum

$F_H = \rho C_p C_H |\underline{V}_a|(T_w - T_a)$ sensible heat → W/m^2

$F_q = \rho L_q C_q |\underline{V}_a|(q_w - q_a)$ latent heat → W/m^2

$F_M = \rho C_D |\underline{V}_a| \underline{V}_a$ momentum → $dynes/cm^2$

where, subscript a → anemometer level.
" w → surface

exchange coefficients (dimensionless)

$C_H = 1.4\times10^{-3}$, $C_q = 1.6\times10^{-3}$, $C_D = 1.1\times10^{-3}$ ↳ drag coefficient

$\rho = 1.23\times10^{-3}\, g\, cm^{-3}$

• The roughness parameter Z_0

Over oceans

Charnock formula

$$Z_0 = M\frac{u^{*2}}{g}$$

where M = a constant (≈ 0.04)

u^* = friction velocity

$\left(u^{*2} = -\overline{u'w'} = \frac{|\tau_0|}{\rho_0}\right)$ ← surface stress

Over land

$Z_0 = 15 + (473.6 + 0.03684h)\times10^{-6}$ → unit (cm)

• Surface similarity theory

nondimensionalized vertical gradients of "large-scale" quantities such as wind, potential temperature, and specific humidity can be expressed as universal functions of a nondimensional height z/L.

where L: Monin–Obukhov length

$$\boxed{L = \frac{u^{*2}}{\kappa\beta\theta^*}}$$ → roughly speaking, the ratio of momentum flux to heat flux

where $\beta = \frac{g}{\theta_0}$ —— ① ; κ: von Karman constant

- Nondimensional shear

$$\frac{\kappa z}{u^*}\frac{\partial \bar{u}}{\partial z} = \phi_m(z/L)$$

- Nondimensional vertical gradient of potential temperature

$$\frac{\kappa z}{\theta^*}\frac{\partial \bar{\theta}}{\partial z} = \phi_h(z/h)$$

- Nondimensional vertical gradient of specific humidity

$$\frac{\kappa z}{q^*}\frac{\partial \bar{q}}{\partial z} = \phi_q(z/h)$$

Here
$u^{*2} = -\overline{u'w'}|_0$
$u^*\theta^* = \overline{\theta' w'}|_0$
$u^* q^* = \overline{q' w'}|_0$

Stability: L (Monin–Obukhov length) or Ri_B (bulk Richardson number)

unstable (heat flux up) : $L<0$, $Ri_B<0$
stable (heat flux down) : $L>0$, $Ri_B>0$
neutral (heat flux = 0) : $L=0$, $Ri_B=0$

→ bulk Richardson number

$$\boxed{Ri_B = \beta\frac{\frac{d\bar{\theta}}{dz}}{\left(\frac{du}{dz}\right)^2}}$$

→ unstable case

$$\frac{\kappa z}{u^*}\frac{\partial \bar{u}}{\partial z} = \left(1 - 15\frac{z}{L}\right)^{-1/4}$$

$$\frac{\kappa z}{\theta^*}\frac{\partial \bar{\theta}}{\partial z} = 0.74\left(1 - 9\frac{z}{L}\right)^{-1/6}$$

$$\frac{\kappa z}{q^*}\frac{\partial \bar{q}}{\partial z} = 0.74\left(1 - 9\frac{z}{L}\right)^{-1/6}$$

by least squares fit

→ stable case

$$\frac{\kappa z}{u^*}\frac{\partial \bar{u}}{\partial z} = 1.0 + 4.7\frac{z}{L}$$

$$\frac{\kappa z}{\theta^*}\frac{\partial \bar{\theta}}{\partial z} = 0.74 + 4.7\frac{z}{L}$$

$$\frac{\kappa z}{q^*}\frac{\partial \bar{q}}{\partial z} = 0.74 + 4.7\frac{z}{L}$$

From ① → $\frac{z}{L} = \frac{z\kappa\beta\theta^*}{u^{*2}}$ —— ⊛

Here, the four eqs (for the stable or the unstable case),
four unknowns (u^*, θ^*, q^*, L)

⇒ surface fluxes of momentum, heat, and moisture.

☆
$F_M = u^{*2} = -\overline{u'w'}|_0$
$F_H = -u^*\theta^* = \overline{\theta' w'}|_0$
$F_q = -u^* q^* = \overline{q' w'}|_0$

• **Vertical disposition of surface fluxes**

K-theory

The eddy diffusion coefficient for heat and moisture

$$K_H = K_q = \ell^2\frac{\partial|\underline{V}|}{\partial z}F_h Ri_B$$ where ℓ: mixing length.

for momentum,

$$K_M = \ell^2\frac{\partial|\underline{V}|}{\partial z}F_m Ri_B$$

Here $\ell = \frac{\kappa z}{1 + \kappa z/\lambda}$ where κ: von Karman constant ($=0.35$)
λ: asymptotic mixing length
($=450$m for heat + moisture, $=150$m for mom)

$F_h = F_m = \frac{1}{(1+5Ri_B)^2}$, $Ri_B \ge 0$ for stable

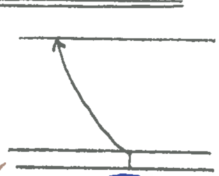
$F_h = F_h(Ri_B)$
$F_m = F_m(Ri_B)$, $Ri_B < 0$ for unstable

The time tendency due to diffusive fluxes at any level.

$$\frac{\partial \tau}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho K\frac{\partial \tau}{\partial z}\right)$$ where $\tau = u, v, \theta$ or q

↓ in σ-coord

$$\frac{\partial \tau}{\partial t} = \frac{g^2}{P_*^2}\frac{\partial}{\partial \sigma}\left(\rho^2 K\frac{\partial \tau}{\partial \sigma}\right)$$



top of the atmosphere

} constant flux layer — roughly 50 meters thick

Goal: $-g\frac{\partial F}{\partial p}$: convergence of flux

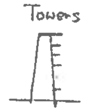
where F is flux of { sensible heat or latent heat or

Formula for F

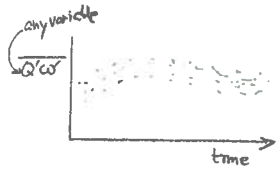
Surface similarity theory → constant flux layer

- In Kansas wheatfield

Towers



covariance of $\overline{\omega' T'}$, $\overline{\omega' q'}$, $\overline{\omega' u'}$, $\overline{\omega' v'}$

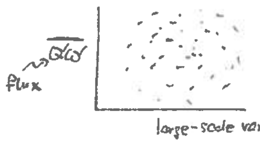


Time series at different levels

large scale data : $\bar{\theta}, \bar{q}, \bar{u}, \bar{v}$

→ Seek a parameterization

Express $\overline{Q'\omega'} = f$ (large scale variables)



→ 1969 Monin-Obukhov, Businger

L : length-scale , Monin-Obukhov length

z : height above ground (earth's surface)

$\frac{z}{L}$: dimensionless (similarity)

abscissa of one diagram

$\frac{L}{P}\frac{\partial Q}{\partial z}$: vertical gradient of any large scale variable such as $\bar{\theta}, \bar{u}, \bar{q}$ etc.

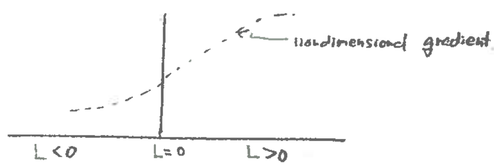
and P is used to nondimensionalize

ordinate

→ L, P unknowns

- Stability

		L
heat flux is up :	unstable	< 0
〃 〃 〃 down :	stable	> 0
neither up or down :	neutral	= 0



$$L = \frac{u^{*2}}{\kappa \beta \theta^*} \longrightarrow L, u^*, \theta^* \text{ —— (a)}$$

$$\frac{\kappa z}{u^*}\frac{\partial \bar{u}}{\partial z} = \left(1 - 15\frac{z}{L}\right)^{-1/4} \longrightarrow u^*, L \text{ —— (b)}$$

$$\frac{\kappa z}{\theta^*}\frac{\partial \bar{\theta}}{\partial z} = 0.74\left(1 - 9\frac{z}{L}\right)^{-1/2} \longrightarrow L, \theta^* \text{ —— (c)}$$

→ 3 eqs, 3 unknowns

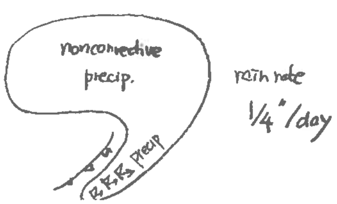
guess on L

find u^*((b)), θ^*((c)) | do loop : difference (small?)

find L ((a))

★ Large-Scale Condensation

non-convective precip. → important for predictive processes



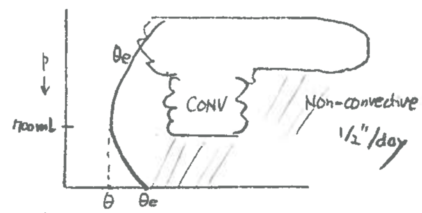
rain rate 1/4"/day

	convective	non convective
Stability	γ_d, γ, γ_m derive!	γ_d, γ_m, γ derive!
Stability	$-\frac{\partial\theta}{\partial p} > 0$, $-\frac{\partial\theta_e}{\partial p} < 0$ (conditionally unstable)	$-\frac{\partial\theta}{\partial p} > 0$, $-\frac{\partial\theta_e}{\partial p} > 0$ (absolutely stable)
ω	$\omega(P_B) < 0$ ↑ PBL	$\omega(P_R) < 0$ ↑ reference
R.H.	$q/q_s(P_B) \geq 0.8$	$q/q_s \approx 1.0$ relax (0.8)

→ paradox (release of latent heating) cooled the atm

→ another name

dynamic ascent of absolute stable saturated air



→ another definition : stratiform rain

How do you compute {nonconvective rain, associated heating, 〃 temp. change, 〃 humidity change

- method 1 : disposition of supersaturation

$$C_p \frac{T}{\theta}\frac{\partial\theta}{\partial t} = +\frac{L\Delta q}{\Delta t} \qquad \frac{\partial q}{\partial t} = -\frac{\Delta q}{\Delta t}$$

How do you get q_s → Teten's formula.

$$e_s = 6.11 \exp\left(\frac{a(T - 273.16)}{T - b}\right)$$
$$q_s = \frac{0.622 e_s}{p}$$

	Saturation over water	Saturation over ice
a	17.26	21.87
b	35.86	7.66

$$H_L = \frac{L\Delta q}{\Delta t}$$

$$R = \frac{1}{g}\int_0^{P_s} \frac{\frac{L\Delta q}{\Delta t}}{L}\, dp$$

- method 2 Dynamic ascent of saturated stable air

$$H_{NC} = -L\frac{dq_s}{dt} \approx +L\omega\frac{\partial q_s}{\partial p}$$

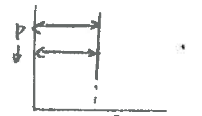
$\frac{\partial q_s}{\partial p}$ is calculated along a condensate i.e. along a moist adiabat

→ difficult one.

along a moist adiabat

θ_e is conserved

or $gz + C_pT + Lq_s$ is conserved



$gz + C_pT + Lq_s$ → moist static energy

$$E_m = gz + C_pT + Lq_s$$

$$\frac{\partial}{\partial p}E_m = g\frac{\partial z}{\partial p} + C_p\frac{\partial T}{\partial p} + L\frac{\partial q_s}{\partial p} = 0 \longleftarrow \text{along a moist adiabat}$$

$$\hookrightarrow -\frac{RT}{p}(1 + 0.61q_s) + C_p\frac{\partial T}{\partial p} + L\frac{\partial q_s}{\partial p} = 0 \text{ —— ①}$$

one eq. has two unknowns $\left(\frac{\partial T}{\partial p}, \frac{\partial q_s}{\partial p}\right)$

※ T, q_s are known

Tetens

$$e_s = 6.11 * e^{\frac{a(T-273.16)}{T-b}}$$

$$q_s = \frac{0.622 e_s}{P - 0.378 e_s}$$

given T and P, Tetens gives you e_s, and q_s

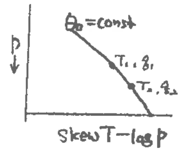
$$\frac{\partial q_s}{\partial p} = \frac{0.622 \frac{\partial e_s}{\partial p}}{p-0.378e_s} - \frac{0.622 e_s}{(p-0.378e_s)^2}\left(1-0.378\frac{\partial e_s}{\partial p}\right)$$

$$\frac{1}{e_s}\frac{\partial e_s}{\partial p} = \frac{a\frac{\partial T}{\partial p}}{T-b} - \frac{a(T-273.16)}{(T-b)^2}\frac{\partial T}{\partial p}$$

⇓

$$A\frac{\partial T}{\partial p} + B\frac{\partial q_s}{\partial p} = C \quad \text{——— (2)}$$

goal → to get $\frac{\partial q_s}{\partial p}$



→ Apply this only if dynamic ascent of absolutely stable saturated air

$$H_L = +L\omega\frac{\partial q_s}{\partial p}$$

you need ω + $\frac{\partial q_s}{\partial p}$

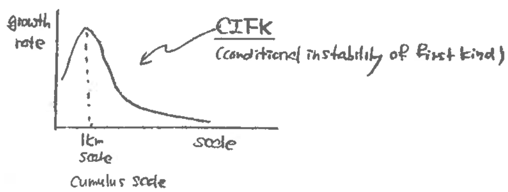
$$R_{NC} = \frac{1}{g}\int_{P_T}^{P_s}\frac{H_L}{L}dp \longrightarrow \text{mm/day}$$

(3) Cumulus parameterization (Krish's note)

Convective heating + convective rain

Conditionally unstable atmosphere

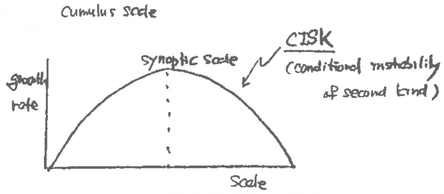
$$H = -L\omega\frac{\partial q_s}{\partial p}$$



↓

$$H = -L\omega_R\frac{\partial q_s}{\partial p}$$

reference (top of PBL)



• Classical Kuo scheme

Conditions to be satisfied for invoking convection

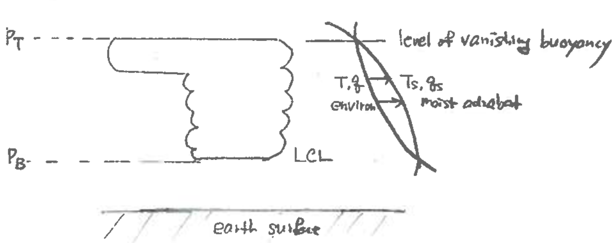
i) conditionally unstable $-\frac{\partial \theta}{\partial p} > 0$; $-\frac{\partial \theta_e}{\partial p} < 0$

ii) net supply of moisture in that regions.

$$I = -\frac{1}{g}\int\left(\nabla\cdot qV + \frac{\partial}{\partial p}q\omega\right)dp > 0$$

Q = max. amount of moisture to form a Kuo cloud over the whole grid.

Kuo cloud



$$Q = \frac{1}{g}\int_{P_T}^{P_B}\frac{C_p(T_s-T)}{L\Delta\tau}dp + \frac{1}{g}\int_{P_T}^{P_B}\frac{q_s-q}{\Delta\tau}dp$$

↳ Cloud time scale

→ too much moistening?
too less heating or rainfall?

$a = \frac{I}{Q}$ fraction area of a grid squares over which you can see Kuo cloud.

* easterly waves $a \approx 2\%$
hurricane $a \approx 40\%$
Tropical depression $a \approx 10\%$

✓ $H_c = aC_p\frac{T_s-T}{\Delta\tau}$ convective heating rate.

✓ $R_c = \frac{1}{g}\int_{P_T}^{P_B}\frac{H_c}{L}dp$

• Any cumulus-param. scheme must provide
- heating profile
- moistening profile
- rainfall rates.

• Big model

Thermodynamic eq.

$$\frac{\partial\theta}{\partial t} + V\cdot\nabla\theta + \omega\frac{\partial\theta}{\partial p} = \frac{1}{C_p}\left(\frac{P_0}{p}\right)^{R/C_p}\sum_i H_i + D_\theta$$

diffusion; $\sum_i H_i$ ↑ $H_c + H_x$ ← others

$$= \frac{1}{C_p}\left(\frac{P_0}{p}\right)^{R/C_p}\frac{aC_p(T_s-T)}{\Delta\tau} + H_x + D_\theta$$

moisture eq.

$$\frac{\partial q}{\partial t} + V\cdot\nabla q + \omega\frac{\partial q}{\partial p} = E - P$$

evaporation, precipitation

$$\boxed{\frac{\partial q}{\partial t} = \frac{a(q_s-q)}{\Delta\tau}}$$

$$\therefore \frac{\partial q}{\partial t} = -V\cdot q - \omega\frac{\partial q}{\partial p} + E - P$$

$$= -\nabla\cdot qV - \frac{\partial}{\partial p}q\omega + E - P$$

$$+\frac{1}{g}\int_{P_T}^{P_B}(\quad)dp.$$

$$+\frac{1}{g}\int_{P_T}^{P_B}\frac{\partial q}{\partial t}dp = I - R \quad \text{precip.}$$

$$\boxed{R = \frac{1}{g}\int_{P_T}^{P_B}a\frac{C_p(T_s-T)}{L\Delta\tau}dp}$$

$$\frac{1}{g}\int_{P_T}^{P_B}\frac{\partial q}{\partial t}dp = aQ - R$$

$$= \frac{a}{g}\int_{P_T}^{P_B}\cancel{\frac{C_p(T_s-T)}{L\Delta\tau}}dp + \frac{a}{g}\int_{P_T}^{P_B}\frac{q_s-q}{\Delta\tau}dp - \cancel{\frac{a}{g}\int_{P_T}^{P_B}\frac{C_p(T_s-T)}{L\Delta\tau}dp}$$

$$\boxed{\int_{P_T}^{P_B}\frac{\partial q}{\partial t}dp = \int_{P_T}^{P_B}\frac{a(q_s-q)}{\Delta\tau}dp}$$

$$\therefore \frac{\partial q}{\partial t} = \frac{a(q_s-q)}{\Delta\tau} + C$$ → vertical integral has to be zero!

(3) Cumulus parameterization (Text)

• The scale of cumulus clouds is much smaller than the scale of the grid squares (or the smallest resolvable scale) of a numerical weather prediction model.

• Cumulus parameterization addresses the effects of the cumulus scale on the resolvable scale, given the information on the latter scales.

• 3 parameters
① the vertical distribution of heating
② " " " " moistening
③ the rain rates

• Kuo's scheme

organized cumulus convection requires
- conditional instability.
- net supply of moisture

moisture eq.

fraction area

$$* \quad \frac{\partial q}{\partial t} = a\frac{q_s-q}{\Delta\tau}$$

↳ Cloud time scale parameter

$$a = \frac{-\frac{1}{g}\int_{P_T}^{P_B}\left(\nabla\cdot q\mathbf{V} + \frac{\partial}{\partial p}q\omega\right)dp}{\frac{1}{g}\int_{P_T}^{P_B}\left(\frac{C_p(T_s-T)}{L\Delta\tau} + \frac{q_s-q}{\Delta\tau}\right)dp} = \frac{I}{Q}$$

where Q = amount of moisture supply needed to cover the entire grid-square area by a model cloud
I = available moisture supply (moisture convergence)

Total convective rainfall rate

$$P_T = \frac{1}{g}\int_{P_T}^{P_B}\frac{aC_p(T_s-T)}{L\Delta\tau}dp$$

∵ moisture conservation law

$$\frac{\partial q}{\partial t} = -\nabla\cdot q\mathbf{V} - \frac{\partial}{\partial p}q\omega + E - P$$

assume that evaporation E occurs only at the air-sea interface

$$\frac{1}{g}\int_{P_T}^{P_B}\frac{\partial q}{\partial t}dp = I - P_T$$

$$\because I = aQ = a\left(\frac{1}{g}\int_{P_T}^{P_B}C_p\frac{T_s-T}{L\Delta\tau}dp + \frac{1}{g}\int_{P_T}^{P_B}\frac{q_s-q}{\Delta\tau}dp\right)$$

$$\therefore \frac{1}{g}\int_{P_T}^{P_B}\frac{\partial q}{\partial t}dp = \frac{a}{g}\int_{P_T}^{P_B}\frac{q_s-q}{\Delta\tau}dp$$

$I = I_q + I_\theta$ → condensational heating
I_q → raising the level of moisture

→ modified Kuo scheme

Supply of moisture ※ ignore horizontal advection

$$\boxed{I_L = \frac{1}{g}\int_{P_T}^{P_B}\omega\frac{\partial q}{\partial p}dp}$$: known from large-scale

where P_T : cloud top → vanishing buoyancy level, P_B : cloud base (LCL)

not enough

mesoscale convergence parameter (η)
moistening parameter (b)

→ total moisture supply

$$I = I_L(1+\eta)$$

→ I is partitioned into a precip. part (R) and a moistening part

$$R = I(1-b) = I_L(1+\eta)(1-b) = I_\theta$$
$$M = Ib = I_L(1+\eta)b = I_q$$

→ adiabatic cooling added.

$$Q = \frac{1}{g}\int_{P_T}^{P_B}\frac{q_s-q}{\Delta\tau}dp + \frac{1}{g}\int_{P_T}^{P_B}\left(\frac{C_pT(\theta_s-\theta)}{L\theta\Delta\tau} + \boxed{\omega\frac{C_pT}{L\theta}\frac{\partial\theta}{\partial p}}\right)dp$$
$$= Q_q + Q_\theta$$

The temp. & moisture eqs

$$\frac{\partial\theta}{\partial t} + \mathbf{V}\cdot\nabla\theta + \omega\frac{\partial\theta}{\partial p} = a_\theta\left(\frac{\theta_s-\theta}{\Delta\tau} + \omega\frac{\partial\theta}{\partial p}\right)$$
$$\frac{\partial q}{\partial t} + \mathbf{V}\cdot\nabla q = a_q\frac{q_s-q}{\Delta\tau}$$

where
$$a_\theta = \frac{I_\theta}{Q_\theta} = \frac{I(1-b)}{Q_\theta} = \frac{I_L(1+\eta)(1-b)}{Q_\theta} = \frac{R}{Q_\theta}$$
$$a_q = \frac{I_q}{Q_q} = \frac{Ib}{Q_q} = \frac{I_Lb(1+\eta)}{Q_q} = \frac{M}{Q_q}$$

If b + η determined → a_θ, a_q. ※(Q_θ, Q_q are knowns)

a closure for b and η → a screening multiregression analysis.

$$\frac{M}{I_L} = a_1\zeta + b_1\bar{\omega} + C_1 = b(1+\eta)$$
$$\frac{R}{I_L} = a_2\zeta + b_2\bar{\omega} + C_2 = (1+\eta)(1-b)$$

where ζ = relative vorticity of 700mb
$\bar{\omega}$ = vertically averaged vertical velocity

Apparent heat source Q_1, apparent moisture sink Q_2

$$Q_1 = a_\theta\left(C_p\frac{T}{\theta}\frac{\theta_s-\theta}{\Delta\tau} + \omega C_p\frac{T}{\theta}\frac{\partial\theta}{\partial p}\right) + C_p\frac{T}{\theta}(H_R+H_s)$$
$$Q_2 = -La_q\left(\frac{q_s-q}{\Delta\tau} + \omega\frac{\partial q}{\partial p}\right) = -L\left[a_q\frac{q_s-q}{\Delta\tau} + \omega\frac{\partial q}{\partial p}\right]$$

Total rain

$$P = \frac{1}{g}\int_{P_T}^{P_B}a_\theta C_p\frac{T}{\theta}\frac{\theta_s-\theta}{L\Delta\tau}dp$$

- Derive Q_1 and Q_2 based on the modified Kuo scheme

dry static energy is defined as

$$S = gz + C_pT$$
$$dS = d(gz + C_pT) \quad \because dp = -\rho g dz$$
$$= C_pdT - \alpha dp$$
$$\frac{dS}{T} = C_p d\ln T - \frac{1}{\rho T}dp \quad \because p = \rho RT$$
$$= C_p d\ln T - R d\ln p \quad \text{——— ①}$$

From the Poisson's eq $\theta = T\left(\frac{P_0}{P}\right)^{R/C_p}$

$$C_p d\ln\theta = C_p d\ln T - R d\ln p \quad \text{——— ②}$$

Comparing ① and ②,

$$C_p d\ln\theta = \frac{dS}{T} \qquad \therefore dS = C_pT\frac{d\theta}{\theta}$$

So, other heating terms

$$Q_1 = \frac{\partial S}{\partial t} + \mathbf{V}\cdot\nabla S + \omega\frac{\partial S}{\partial p} = C_p\frac{T}{\theta}\left(\frac{\partial\theta}{\partial t} + \mathbf{V}\cdot\nabla\theta + \omega\frac{\partial\theta}{\partial p} + \boxed{H_R+H_s}\right)$$

using the thermodynamic eq. for the modified Kuo scheme

$$Q_1 = C_p\frac{T}{\theta}a_\theta\left(\frac{\theta_s-\theta}{\Delta\tau} + \omega\frac{\partial\theta}{\partial p}\right) + C_p\frac{T}{\theta}(H_R+H_s)$$
$$= a_\theta\left(C_p\frac{T}{\theta}\frac{\theta_s-\theta}{\Delta\tau} + \omega C_p\frac{T}{\theta}\frac{\partial\theta}{\partial p}\right) + C_p\frac{T}{\theta}(H_R+H_s)$$

From the apparent moisture sink

$$Q_2 = -L\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla + \omega\frac{\partial}{\partial p}\right)q$$

using the moisture eq. $\frac{\partial q}{\partial t} + \mathbf{V}\cdot\nabla q = a_q\frac{q_s-q}{\Delta\tau}$

$$Q_2 = -L\left(a_q\frac{q_s-q}{\Delta\tau} + \omega\frac{\partial q}{\partial p}\right)$$

- The Arakawa-Schubert cumulus parameterization

→ look at other notes.

- Radiation

- Solar constant : $S_0 = 1380\ W/m^2$
- Zenith angle : ζ

$$\cos\zeta = \sin\phi\sin\delta + \cos\phi\cos\delta\cos h$$

ϕ = latitude
δ = declination of the sun
h = hour angle

- optical depth : $\tau = k_a u$

∵ k_a = absorption coefficient (cm^2/g)
u = optical path (g/cm^2)

$$= -\frac{1}{g}\int_{P_1}^{P_2}q_z dp > 0$$

P_2 → upper, P_1 → lower, q_z → specific humidity.

- optical path length

$$W = -\frac{1}{g}\int_{P_2}^{P_1}q\left(\frac{P}{P_0}\right)^{0.85}\left(\frac{T_0}{T}\right)^{1/2}dp$$

∵ P_0 = 1013.25mb, T_0 = 273.15K

→ reference level

→ the depth that the accumulated optical path would extend at S.T.P.

- Heating rate due to SW radiative flux

$$\frac{dT}{dt} = -\frac{g}{C_p}\underbrace{\boxed{\frac{d(S^{\downarrow}-S^{\uparrow})}{dp}}_{<0}}_{\text{flux convergence}} > 0$$

- Cooling rate due to LW radiative flux

$$\frac{dT}{dt} = -\frac{g}{C_p}\underbrace{\boxed{\frac{d(F^{\downarrow}-F^{\uparrow})}{dp}}}_{>0} < 0$$

- Cloud fraction

$$C_{L,M,H} = \left(\frac{\overline{RH}_{L,M,H} - RH_c}{1.0 - RH_c}\right)^2$$

$\left\{\begin{array}{l}\overline{RH} = \text{mean } RH \text{ in a layer}\\ RH_c = \text{threshold } RH\end{array}\right.$

- General form of the radiative transfer eqs (RTE)

$$-\frac{\mu dF}{d\tau} = -F + J \quad \text{source function}$$

$J \to B(T) = \sigma T^4$

σ: stefan-Boltzman constant $= 5.67\times10^{-8}\ Wm^{-2}K^{-4}$

$\because \mu \equiv \cos\zeta$

→ Schwarzchild's eq.

$$-\frac{\mu dF}{d\tau} = -F + \sigma T^4$$

$$\xrightarrow{\text{Sol}} F(z) = \underset{\text{surface}}{F(z_s)} e^{-\tau/\mu} + \int_{\tau}^{\tau_s} \sigma T^4 e^{-\tau/\mu}\frac{d\tau}{\mu}$$

→ BBL (Beer-Bouger-Lambert) eq.

$$-\frac{\mu dF}{d\tau} = -F$$

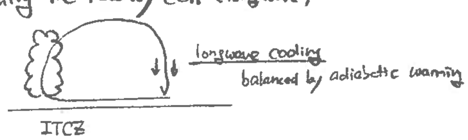
$$\longrightarrow F(z) = F(z_s)e^{-\tau/\mu}$$

- absorptivity/Emissivity method

$\left\{\begin{array}{l}\text{SW : absorptivity}\\ \text{LW : emissivity}\end{array}\right.$

- Radiative applications in NWP (some examples)

① modeling the Hadley cell (longwave)



② Tropical, subtropical high ← H)

③ Land area heat lows ⟶

④ Conditional instability restoration in the Hurricane environment.

⑤ Radiative effects of dust and aerosols

⑥ modeling of the diurnal change.

⑦ Modeling the African waves

⑧ Modeling of the monsoon

⑨ Modeling of the Siberian high : during winter time (longwave)

⑩ Cloud radiative effects from large sheets of altostratus

⑪ Cloud radiative effects from large sheets of coastal stratus.

END