* Black-body radiation

MET4450/450 (Black-body radiation): Ahlquist (1 hour, 1997) *

• Question

(a) What is the formula for the flux density in watts per square meter emitted by a blackbody at temperature T? What is the name of the constant in this formula?

(b) According to Kirchhoff's law, how are absorptivity and emissivity related for a body in thermodynamic equilibrium?

(c) What is the absorptivity of a black body? What is the emissivity of a black body?

(d) Consider a simple radiative model in which the atmosphere is treated as N isothermal layers, each of which absorbs and emits as a black body at the top and bottom of the layer. Let T_i represent the temperature of the i-th layer. Let T_0 represent the temperature of the surface, which is also nearly a black body in the infrared. Assuming steady state, write the flux balance equation for the i-th atmospheric layer where $i = 1, \ldots, N$-1.

(e) What is the flux balance equation for the N-th layer whose top is at outer space?

a) $F = \sigma T^4$

where σ is the Stefan-Boltzmann constant.

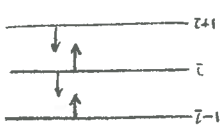
b) A body in thermodynamic equilibrium, its absorptivity equals emissivity at all wavelengths

$$A_\lambda = \varepsilon_\lambda$$

c) Both the absorptivity and emissivity of a blackbody are unit, 1.

$$A_\lambda = \varepsilon_\lambda = 1$$

d)



In the steady state, layer i should emits the same amount of energy as it receives

the upward emits $F_i\uparrow = \sigma T_i^4$

the downward emits $F_i\downarrow = \sigma T_i^4$

At the same time, it receives radiation from layer i+1 and layer i-1

from layer i+1 : $F_{i+1}\downarrow = \sigma T_{i+1}^4$

" " i-1 : $F_{i-1}\uparrow = \sigma T_{i-1}^4$

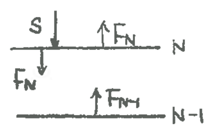
At steady state,

$$F_i\uparrow + F_i\downarrow = F_{i+1}\downarrow + F_{i-1}\uparrow$$

$$\therefore \sigma T_i^4 + \sigma T_i^4 = \sigma T_{i+1}^4 + \sigma T_{i-1}^4$$

$$\Rightarrow T_i^4 = \frac{1}{2}(T_{i-1}^4 + T_{i+1}^4)$$

e)



The N-th layer receives energy from the sun, the flux density S, and from layer N-1, $F_{N-1}\uparrow = \sigma T_{N-1}^4$ at the same time. It emits energy upward and downward : $F_N\uparrow = F_N\downarrow = \sigma T_N^4$

$\therefore$ Steady state : $S + F_{N-1}\uparrow = F_N\uparrow + F_N\downarrow$

$$\Rightarrow S + \sigma T_{N-1}^4 = 2\sigma T_N^4$$

Comment : For a somewhat more realistic model for the earth's atmosphere, one can treat the atmosphere as being a black body in the infrared and transparent in the visible.

1/1

* Vorticity eq.

MET4301/5311(Dynamics I): Ahlquist (60 minutes)

vertical adv.

β effect.

- The equation for the vertical component of vorticity, $\zeta = \partial v/\partial x - \partial u/\partial y$, is

$$\frac{\partial \zeta}{\partial t} = -\mathbf{V} \bullet \nabla \zeta - w\frac{\partial \zeta}{\partial z} - v\frac{\partial f}{\partial y} - (f+\zeta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

horiz advection

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

where $f = 2\Omega \sin\varphi$ and where the last term in the vorticity equation is a rearranged version of

$$\frac{\partial}{\partial x}\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

Explain physically the way in which each of the 6 terms on the right hand side caused vorticity to change. For each term, your written explanation should include a drawing of a physical example in which that term is important. Examples from meteorology or oceanography are preferable, but any examples involving the flow of a liquid or gas are acceptable. You can have 6 different examples if you want, or you may know of some examples where several of the terms on the right hand side are important. Just make sure that you give a physical example for each term.

* look at the Holton's note

$$\frac{d}{dt}(\zeta+f) = \underbrace{-(\zeta+f)\nabla\cdot\mathbf{V}}_{\text{ⓐ divergence}} \underbrace{-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}_{\text{ⓑ tilting term}} + \underbrace{\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}_{\text{ⓒ Solenoidal term}}$$

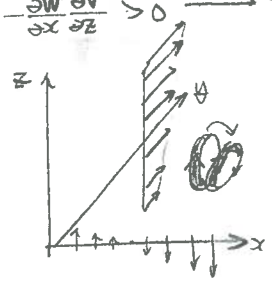
ⓐ divergence term

if there is convergence

$-\nabla\cdot\mathbf{V} > 0 \Rightarrow \therefore \frac{d\zeta}{dt} > 0$

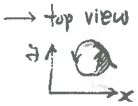
ⓑ tilting term

$-\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} > 0 \longrightarrow \frac{d\zeta}{dt} > 0$



Let's consider $-\frac{\partial w}{\partial x} > 0$, $\frac{\partial v}{\partial z} > 0$

→ top view



ⓒ Solenoidal term

$\because \nabla p \cdot d\boldsymbol{l} = dp$

$$\oint -\alpha dp = -\oint \alpha \nabla p \cdot d\boldsymbol{l}$$

← Stokes' theorem

$$= -\iint_A \nabla \times (\alpha \nabla p)\cdot \mathbf{k}\, dA$$

$$= -\iint_A (\nabla\alpha \times \nabla p)\cdot \mathbf{k}\, dA$$

$$= -\iint_A \left(\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}\right) dA$$

$$= \iint_A \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) dA.$$

Sound waves

MET4302/5312(Dynamics II): Ahlquist (60 minutes) *

- Question

(a) Give an example of a process in meteorology that involves sound waves.

(b) Derive the linearized equations that describe how one-dimensional sound waves propagate. Solve for the wave speed, and put in numbers to get an order of magnitude value for the wave speed.

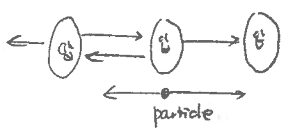
(c) How can sound waves be eliminated from numerical models?

- look at my Holton's note (Sec. 7.3.1)

Sound waves = Acoustic waves. → longitudinal waves

→ particle oscillations are parallel to the direction of propagation.

→ restoring force : adiabatic compression/expansion



Let's consider the sound wave which only propagates x-direction

$$\begin{cases} u = u(x,t) \\ v = 0 = w \\ \frac{d}{dt} = \frac{\partial}{\partial x} + u\frac{\partial}{\partial x} \end{cases}$$

→ x-component momentum eq.

$$\ast\ \frac{du}{dt} - \cancel{fv} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \cancel{F_r} \quad \text{——— ⓐ}$$

→ conti. eq.

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla\cdot\mathbf{V} \rightarrow \frac{1}{\rho}\frac{d\rho}{dt} = -\frac{\partial u}{\partial x} \quad \text{——— ⓑ}$$

→ thermodynamic eq.

$$\frac{ds}{dt} = \frac{\dot{q}}{T} = c_p\frac{d\ln\theta}{dt} \rightarrow \frac{d\ln\theta}{dt} = 0 \quad \text{——— ⓒ}$$

adiabatic

∵ Poisson's eq.

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p} = \frac{p}{\rho R}\left(\frac{p_0}{p}\right)^{R/c_p}$$

$$\ln\theta = \ln p - \ln\rho - \ln R + \ln p_0^{R/c_p} - \frac{R}{c_p}\ln p$$

$$\frac{d\ln\theta}{dt} = \frac{d\ln p}{dt} - \frac{d\ln\rho}{dt} - \frac{R}{c_p}\frac{d\ln p}{dt}$$

$$= \left(1 - \frac{R}{c_p}\right)\frac{d\ln p}{dt} - \frac{d\ln\rho}{dt} = 0$$

$$\because 1 - \frac{R}{c_p} = \frac{c_p - R}{c_p} = \frac{c_v}{c_p} = \frac{1}{\gamma}$$

ⓒ →

$$\frac{1}{\gamma}\frac{d}{dt}\ln p - \frac{d\ln\rho}{dt} = 0 \quad \text{——— ⓓ}$$

$$\because \gamma = \frac{c_p}{c_v}$$

From ⓑ + ⓓ, remove ρ

$$\ast\ \frac{1}{\gamma}\frac{d}{dt}\ln p + \frac{\partial u}{\partial x} = 0 \quad \text{——— ⓔ}$$

Let's linearize eqs ⓐ + ⓔ

$$\because \begin{cases} u(x,t) = \bar{u} + u'(x,t) \\ p(x,t) = \bar{p} + p'(x,t) \\ \rho(x,t) = \bar{\rho} + \rho'(x,t) \end{cases}$$

$$\because \left(\frac{1}{\rho} = \frac{1}{\bar\rho + \rho'} = \frac{1}{\bar\rho}\left(1 + \frac{\rho'}{\bar\rho}\right)^{-1} \approx \frac{1}{\bar\rho}\left(1 - \frac{\rho'}{\bar\rho}\right)\right) \quad \because \left|\frac{\rho'}{\bar\rho}\right| \ll 1$$

ⓐ →

$$\underbrace{\frac{\partial}{\partial t}(\bar{u} + u')}_{\frac{\partial u'}{\partial t}} + \underbrace{(\bar{u} + u')\frac{\partial}{\partial x}(\bar{u} + u')}_{\bar{u}\frac{\partial u'}{\partial x}\ (u'\frac{\partial u'}{\partial x}\text{ is small})} + \frac{1}{\bar\rho + \rho'}\frac{\partial}{\partial x}(\bar{p} + p') = 0$$

$$\frac{1}{\bar\rho}\left(1 - \frac{\rho'}{\bar\rho}\right)$$

ⓔ →

$$\frac{\partial}{\partial t}(\cancel{\bar{p}} + p') + (\bar{u} + u')\frac{\partial}{\partial x}(\bar{p} + p') + \gamma(\bar{p} + p')\frac{\partial}{\partial x}(\bar{u} + u') = 0$$

where ($O(\epsilon^2)$ ignored
mean state satisfies original eq

$$\begin{cases} \frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \frac{1}{\bar\rho}\frac{\partial p'}{\partial x} = 0 \quad \text{——— ①} \\ \frac{\partial p'}{\partial t} + \bar{u}\frac{\partial p'}{\partial x} + \gamma\bar{p}\frac{\partial u'}{\partial x} = 0 \quad \text{——— ②} \end{cases}$$

② →

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)p' + \gamma\bar{p}\left(\cancel{\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}}\right)\frac{\partial u'}{\partial x} = 0$$

① →

$$\gamma\bar{p}\left(\cancel{\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}}\right)\frac{\partial u'}{\partial x} + \frac{\gamma\bar{p}}{\bar\rho}\frac{\partial^2 p'}{\partial x^2}:$$

remove u' from ① + ②

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)② - \gamma\bar{p}\frac{\partial}{\partial x}①$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 p' - \frac{\gamma\bar{p}}{\bar\rho}\frac{\partial^2 p'}{\partial x^2} = 0$$

Assume $p' = Ae^{ik(x-ct)} = Ae^{ikx - i\nu t}$ $\quad \because kc = \nu \rightarrow c = \frac{\nu}{k}$

and substituting,

$$\therefore (-ikc + ik\bar{u})^2 - \frac{\gamma\bar{p}}{\bar\rho}(ik)^2 = 0$$

$$\therefore c = \bar{u} \pm \left(\frac{\gamma\bar{p}}{\bar\rho}\right)^{1/2} \quad \text{adiabatic speed of sound.}$$

$$= \bar{u} \pm \sqrt{\gamma R\bar{T}} \qquad \therefore c = c(T)$$

Doppler shifting

★ The assumption of incompressibility is sufficient to exclude sound waves. $\left(\frac{d\rho}{dt} = 0 \rightarrow \text{"sound wave" gone!!}\right)$

In Numerical model,

hydrostatic balance → eliminate vertical sound waves

$\omega|_{z=0} = 0$ → eliminate horizontal sound waves

* Rossby wave (barotropic vorticity eq.)

MET4302/5312(Dynamics II): Ahlquist (1 hour, 1997)

• Question

(a) A simplified version of the linearized nondivergent barotropic vorticity equation can be written as

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial^2\psi}{\partial x^2}+\beta\frac{\partial\psi}{\partial x}=0$$

Where $\bar{u}$ and β are constants. Assume that $\psi(x,t)=\Psi e^{i(kx-\nu t)}$ where Ψ is a constant, and solve for the dispersion relation.

(b) Using the dispersion relation you just found, compute the phase speed. In synoptic meteorology, one hears the statement: Short waves travel faster than long waves. Long waves travel faster than short waves, though, in a reference frame moving with the mean flow (so $\bar{u}=0$). Explain how there is no contradiction in theses two statements.

(c) Compute the group velocity of a Rossby wave.

(d) Suppose that Rossby waves are excited by an isolated disturbance. Explain why the group velocity is *not* of direct use for computing the speed of a Rossby wave packet near the excitation point. Hint: To compute a group velocity, you must evaluate the group velocity formula at some particular wavenumber *k*. What "value" of *k* will be excited by an isolated disturbance?

(e) Can the Rossby wave solution that you found in part (a) ever grow exponentially in time? Why or why not?

(a) Substitute $\psi(x,t)=\Psi e^{i(kx-\nu t)}$ into the eq., we get

$-k^2(-i\nu+ik\bar{u})\Psi e^{i(kx-\nu t)}+\beta ik\Psi e^{i(kx-\nu t)}=0$

$-k^2(k\bar{u}-\nu)+\beta k=0$

$\Rightarrow \nu=k\bar{u}-\frac{\beta}{k}$

(b) phase speed $c=\frac{\nu}{k}=\bar{u}-\frac{\beta}{k^2}$

wave number for shortwave, say, k_s

" " longwave, say, k_L

We know $k_s>k_L$

$\therefore$ from $c=\bar{u}-\beta/k^2 \Rightarrow$ shortwave $C_s>$ longwave C_L

But if we express this statement in a frame that is moving with the mean flow, then the waves move westward relative to the observer with speed $|c|=\beta/k^2$, thus, we can say, since $k_s>k_L$, then the long waves travel faster than short waves.

(c) group velocity $C_g=\frac{\partial\nu}{\partial k}=\bar{u}+\frac{\beta}{k^2}$

So energy moves to the east faster than the mean flow.

(d) If an isolated disturbance is excited, it will contain many wavenumbers, so it is not clear which wavenumber to use when computing C_g. Group velocity is useful often the waves had time to separate and one can assign wavenumbers to different parts of the wave train

(e) The Rossby wave CAN NOT grow exponentially. Since the solution phase speed c is real, it does not have the imaginary part.

From $\psi(x,t)=\Psi e^{ik(x-ct)}$ we know the wave can not grow exponentially

It can only oscillate around its equilibrium positions.

Baroclinic Instability #1

MET4302/5312 (Baroclinic instability): Ahlquist (60 minutes) ***

- Thoroughly discuss baroclinic instability from the perspective of the two-level linearized quasi-geostrophic equations, where phase speed is given by

$$c = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} \pm \delta^{1/2}$$

where

$$\delta \equiv \frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2} - \frac{U_T^2(2\lambda^2-k^2)}{(k^2+2\lambda^2)}$$

$$U_m \equiv \frac{1}{2}(U_1+U_3)$$

$$U_T \equiv \frac{1}{2}(U_1-U_3)$$

$$\lambda^2 \equiv f_0^2 / \left[\sigma(\Delta p)^2\right]$$

and where U_1 and U_3 are the upper and lower level mean zonal winds or ocean currents, σ is the static stability (always positive for QG theory), and Δp is the pressure difference between the two layers. In your discussion of baroclinic instability, you should explain:

(a) Are the waves dispersive? Yes. the radius of deformation.

(b) What name is given to λ^{-1}, and what is its significance? (Hint: λ^2 has units of m^2).

(c) What is slantwise (or sloping) convection?

(d) What physical effect helps to stabilize "long" waves? What physical effect helps to stabilize "short" waves? → static stability → β-effect

(e) What role does baroclinic instability play in the Earth's atmospheric energy budget?

As part of your answer, you should make a sketch of a stability diagram. Do this by labeling regions of stability and instability on a parameter space diagram showing $k^2/(2\lambda^2)$ on the abscissa (x-axis) and $2\lambda^2 U_T/\beta$ on the ordinate (y-axis).

look at the attached note.

Baroclinic instability #2

MET4302/5312 (Baroclinic instability): Ahlquist (60 minutes) ***

- Thoroughly discuss baroclinic instability from the perspective of the two-level linearized quasi-geostrophic equations, where phase speed is given by

$$c = U_m - \frac{\beta\left(k^2+\lambda^2\right)}{k^2\left(k^2+2\lambda^2\right)} \pm \delta^{1/2}$$

where
$$\delta \equiv \frac{\beta^2\lambda^4}{k^4\left(k^2+2\lambda^2\right)^2} - \frac{U_T^2\left(2\lambda^2-k^2\right)}{\left(k^2+2\lambda^2\right)}$$

$$U_m \equiv \frac{1}{2}(U_1+U_3)$$

$$U_T \equiv \frac{1}{2}(U_1-U_3)$$

$$\lambda^2 \equiv f_0^2 / \left[\sigma(\Delta p)^2\right]$$

and where U_1 and U_3 are the upper and lower level mean zonal winds or ocean currents, σ is the static stability (always positive for QG theory), and Δp is the pressure difference between the two layers. In your discussion of baroclinic instability, you should explain:

(a) Are the waves dispersive?

(b) What name is given to λ^{-1}, and what is its significance? (Hint: λ^2 has units of m^2).

(c) What is slantwise (or sloping) convection?

(d) What physical effect helps to stabilize "long" waves? What physical effect helps to stabilize "short" waves?

(e) What role does baroclinic instability play in the Earth's atmospheric energy budget?

As part of your answer, you should make a sketch of a stability diagram. Do this by labeling regions of stability and instability on a parameter space diagram showing $k^2/(2\lambda^2)$ on the abscissa (x-axis) and $2\lambda^2 U_T/\beta$ on the ordinate (y-axis).

look at the attached note.

Baroclinic instability #3

MET4302/5312 (Atmospheric dynamics): Ahlquist (60 minutes)

- Thoroughly discuss baroclinic instability from the perspective of the two-level linearized quasi-geostrophic equations, where phase speed is given by

$$c = U_m - \frac{\beta\left(k^2 + \lambda^2\right)}{k^2\left(k^2 + 2\lambda^2\right)} \pm \delta^{1/2}$$

for $\quad k \equiv$ zonal wavenumber $= 2\pi$ /wavelength

$$\delta \equiv \frac{\beta^2\lambda^4}{k^4\left(k^2 + 2\lambda^2\right)^2} - \frac{U_T^2\left(2\lambda^2 - k^2\right)}{\left(k^2 + 2\lambda^2\right)}$$

$$U_m \equiv \frac{1}{2}\left(U_{\text{upper}} + U_{\text{lower}}\right)$$

$$U_T \equiv \frac{1}{2}\left(U_{\text{upper}} - U_{\text{lower}}\right)$$

and $\quad \lambda^2 \equiv f_0^2 / \left(\sigma \Delta p^2\right)$

where U_{lower} and U_{upper} are the basic state flow speeds at 75 kPa and 25 kPa, $\Delta p = 50$ kPa, and σ is the basic state stability at 50 kPa. In your discussion of baroclinic instability, you should explain: (i) What name is given to λ^{-1}, and what is its significance? (ii) What is slantwise (or slopping) convection? (iii) What role does baroclinic instability play in the global atmosphere's energy budget?

look at the attached note.

Baroclinic Instability → Holton's Ch 8.2

- The expression for the phase speed C can be multiplied by k to give

  $$\nu = kC = k(\sim)$$

  This is a dispersion relation. We say that waves are dispersive if their phase speed depends on wavenumber. In this case the phase speed C clearly depends on the wavenumber k. Therefore the waves described by this dispersion relation are dispersive.

- It is clear that if $(\delta < 0)$, the phase speed will have an imaginary part and so a solution of the form $e^{ik(x-ct)}$ will become $e^{ik(x-C_r t)+kC_I t}$. That is, there will be an exponential term in time, $e^{kC_I t}$.
  If $C_I < 0$ the perturbation will exponentially decay; with $C_I > 0$, exponential growth. Thus there exists the possibility for instability in the waves described by this dispersion relation.

- $U_T = 0$

  As the first special case we consider, let $U_T = 0$ so that the basic state thermal wind vanishes and the mean flow is barotropic. With $U_T = 0$, the term $\pm\delta^{1/2}$ becomes $\pm\beta\lambda^2/k^2(k^2+2\lambda^2)$. We have two phase speed,

  $$C_1 = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} + \frac{\beta\lambda^2}{k^2(k^2+2\lambda^2)} = U_m - \frac{\beta \cancel{k^2}}{\cancel{k^2}(k^2+2\lambda^2)}$$

  and

  $$C_2 = U_m - \frac{\beta k^2+\beta\lambda^2}{k^2(k^2+2\lambda^2)} - \frac{\beta\lambda^2}{k^2(k^2+2\lambda^2)} = U_m - \frac{\beta\cancel{(k^2+2\lambda^2)}}{k^2\cancel{(k^2+2\lambda^2)}}$$

  or

  $C_1 = U_m - \beta(k^2+2\lambda^2)^{-1}$ → Internal baroclinic Rossby wave
  $C_2 = U_m - \beta k^{-2}$ → Barotropic Rossby wave

  The phase speeds are real and so the waves are stable and correspond to the free (normal mode) oscillations for the two level model with a barotropic base state current. The phase speed C_2 is the dispersion relation for a barotropic Rossby wave with no y dependence. The perturbation is barotropic in structure. The phase speed C_1 may be interpreted as that for an internal baroclinic Rossby wave. It is also analogous to the phase speed for a homogeneous ocean with a free surface provided we replace $2\lambda^2$ with f_0^2/gH. In each of these cases there is vertical motion associated with the Rossby wave described by C_1. With vertical motion present the static stability can modify the wave speed. For the wave described by C_1, the lower + upper streamfunction fields are 180° out of phase ⇒ the perturbation is baroclinic (though the base state is barotropic). The mid-level ω field is 1/4 wavelength out of phase with the upper level Φ field with max ↑ motion west of the upper level Φ field. This pattern of ω makes physical sense when we note that $C_1 - U_m < 0$ so that the disturbance moves westward relative to the mean flow.

  The relative motion of these waves is

  $C_1 - U_m = -\beta(k^2+2\lambda)^{-1}$ → Internal baroclinic mode
  $C_2 - U_m = -\beta k^{-2}$ → barotropic mode

  From this we see that the baroclinic mode moves westward at a slower rate than the barotropic mode. It has been shown that the internal baroclinic mode is a result of the upper boundary condition → $\omega = 0$ at the top of the model atmosphere. This upper BC effectively puts a lid on the atmosphere. The real atmosphere does not have a free oscillation which we can identify as an internal baroclinic Rossby wave.

- $\beta = 0$

  We now consider a second special case, $\beta = 0$. This is qualitatively similar to the Eady problem. This case corresponds to the f-plane approximation. In a laboratory experiment, this case corresponds to one in which the fluid bounded above + below by rotating horizontal plates so that the gravity + rotation vectors are everywhere parallel. With $\beta = 0$ our eq. for C reduces to

  $$C = U_m \pm U_T\left(\frac{-2\lambda^2+k^2}{k^2+2\lambda^2}\right)^{1/2}$$

  Again we have two modes. Of interest here is the possibility of instability. If the term inside the $(\ )^{1/2}$ is negative we will have an imaginary component to the phase speed. We already shown that the resulting perturbation will grow exponentially if $C_I > 0$ and exponentially decay if $C_I < 0$.
  The condition for instability is that

  $$k^2 < 2\lambda^2 \longrightarrow \text{instability occurs}$$

  Now $k = \frac{2\pi}{L_x}$ so this condition is equivalent to

  ∵ static stability $\left(\sigma = -\frac{\alpha}{\theta}\frac{\partial\theta}{\partial p}\right.$

  $$\frac{4\pi^2}{L_x^2} \le 2\lambda^2$$

  ∵ $\left(\lambda^2 = \frac{f_0^2}{\sigma(\Delta p)^2}\right.$ so $\lambda = \frac{f_0}{\sqrt{\sigma}\,\Delta p}$ and $\frac{1}{\lambda} = \frac{\sqrt{\sigma}\,\Delta p}{f_0}$

  ↓ mks units of 1/s

  $$\frac{2\pi^2}{\lambda^2} \le L_x^2$$

  or

  $$\boxed{L_x \ge \frac{\sqrt{2}\,\pi}{\lambda} = \frac{\sqrt{2\sigma}\,\pi\,\Delta p}{f_0}}$$

  Waves with a wavelength greater than the critical wavelength L_c will be unstable.
  For typical tropospheric conditions

  $\sqrt{2\sigma} \sim 2\times10^{-3}\ m^3/Ns$ $\qquad \sigma \sim 2\times10^{-6}$
  $\Delta p \sim 50\,kPa = 50\times10^3\,Pa$ (N/m^2)
  $f_0 \sim 10^{-4}\,s^{-1}$

  We find

  $$L_c \sim \frac{2\times10^{-3}\frac{m^3}{Ns}\ \pi\ 50\times10^3\frac{N}{m^2}}{10^{-4}\ 1/s} \cong 3141.6\ km \sim 3000\ km$$

  The parameter λ^{-1} has units of m^{-1}. We see that it plays a special role in defining the critical wavelength at which instability occurs. We call λ^{-1} the radius of deformation

  $$\boxed{\lambda^{-1} = \frac{\sigma^{1/2}\Delta p}{f_0} \equiv \frac{\text{effects of stability}}{\text{effects of rotation}} * \text{depth of perturbation}}$$

  This length scale increases with the static stability. In the context of our problem the critical wavelength for instability increases with static stability. This relationship provides insight into the often noted 'short wave stabilization' in baroclinic instability.

  The role of static stability in stabilizing the shorter waves can be understood qualitatively as follows. For a sinusoidal perturbation,

  $$\Phi' = A\exp(ik(x-ct))$$

  The relative vorticity and hence the differential vorticity advection increases as the square of the wavenumber. That is,

  $$\zeta' = \frac{1}{f_0}\nabla^2\Phi' = \frac{A}{f}e^{ik(x-ct)}(-k^2)$$

  $$\zeta' = -\frac{k^2\Phi'}{f_0}$$

  In QG dynamics a secondary vertical circulation serves to maintain hydrostatic temperature changes and geostrophic vorticity changes in the presence of differential vorticity advection.

  ↳ ex. PVA means ζ_g↑. Since $\zeta_g = \frac{1}{f_0}\nabla^2\Phi$, when ζ_g↑, the geopotential Φ↓. In order to produce falling heights rising motion develops.

  Thus, for a geopotential perturbation of fixed amplitude (A), the relative strength of the secondary vertical circulation must increase as the wave length decreases (wavenumber increases). Static stability tends to resist vertical motion. For a given σ, the shortest waves are stabilized. As $\sigma\uparrow$, increasingly longer waves are stabilized.

It is also interest in the $\beta=0$ case to note that the critical wavelength for instability does not depend on the base state mean thermal wind U_T

$$L_c \geq \frac{\sqrt{2}\pi}{\lambda} = \frac{\sqrt{2}\pi}{\left(\frac{4P\sqrt{\sigma}}{f_0}\right)^{-1}} = \frac{\sqrt{2\sigma}\,\pi\,\Delta p}{f_0}$$

$$L_c \geq \sqrt{2}\pi\lambda^{-1}$$

However, the growth rate, α, does depend on the mean thermal wind. To see this suppose that $k^2<2\lambda^2$. Then the phase speed c has both real and imaginary components.

$$c = U_m \pm U_T\left(\frac{k^2-2\lambda^2}{k^2+2\lambda^2}\right)^{1/2} \quad \text{where } k^2<2\lambda^2$$

$$c = U_m \pm i\,\underbrace{U_T\left(\frac{2\lambda^2-k^2}{k^2+2\lambda^2}\right)^{1/2}}_{c_I}$$

For perturbations of the form

$$\psi' = Ae^{ik(x-ct)} = Ae^{ik(x-(c_R+ic_I)t)} = Ae^{ik(x-c_Rt)}\,e^{kc_It}$$

We see that if the phase speed has an imaginary component, then the wave will grow or decay exponentially at the rate $\alpha = kc_I$. In the present case

$$\alpha = kc_I = \pm kU_T\left(\frac{2\lambda^2-k^2}{k^2+2\lambda^2}\right)^{1/2}$$

The growth rate increases (in magnitude) linearly with the thermal wind

• general case

From the given phase speed eq. we noted that the question of instability lays in the δ term. A negative δ implies an unstable wave. A positive δ implies a stable wave. Thus, when $\delta=0$ we have a marginally stable wave. The neutral curve is one which connects all U_T and k for which $\delta=0$. When $\delta=0$,

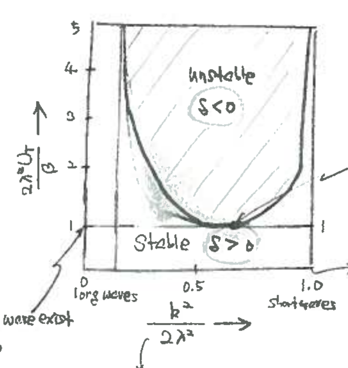
$$\frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2} = \frac{U_T^2(2\lambda^2-k^2)}{k^2+2\lambda^2}$$

This complicated relationship between U_T and k may be meaningfully and most readily displayed by solving the above for $k^4/2\lambda^4$. Doing so yields

$$\frac{k^4}{2\lambda^4} = 1 \pm \left[1-\frac{\beta^2}{4\lambda^4U_T^2}\right]^{1/2}$$

Below we plot the nondimensional quantities $\frac{k^2}{2\lambda^2}$ vs $\frac{2\lambda^2U_T}{\beta}$

measure of zonal wavenumber — proportional to thermal wind



The wavelength of max instability occurs at $k^2=\sqrt{2}\lambda^2$ or $\frac{\sqrt{2}}{2}$ on this diagram.

waves are only unstable where $k^2\leq 2\lambda^2$ or $\frac{k^2}{2\lambda^2}\leq 1$

unstable wave exist only when $\frac{2\lambda^2U_T}{\beta}>1$

proportional to $(\text{wavelength})^{-1}$

$|U_T| > \frac{\beta}{2\lambda^2} \Rightarrow$ The β effect stablizes the long wave end of the spectrum

were shown that at the short wavelengths static stability resists the vertical motion necessary for these waves $\rightarrow$ hence a short wavelength cut off

※ derivation

$$\frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2} = \frac{U_T^2(2\lambda^2-k^2)}{k^2+2\lambda^2}$$

$$\frac{\beta^2}{U_T^2}\frac{\lambda^4}{k^4} = 4\lambda^4-k^4$$

$$\frac{\beta^2}{U_T^2}\frac{\lambda^4}{k^4} = 2\lambda^4\left(2-\frac{k^4}{2\lambda^4}\right) \longrightarrow \frac{\beta^2}{2\lambda^4U_T^2} = \frac{k^4}{\lambda^4}\left(2-\frac{k^4}{2\lambda^4}\right)$$

$$\frac{1}{2}\frac{\beta^2}{2\lambda^4U_T^2} = \frac{k^4}{2\lambda^4}\left(2-\frac{k^4}{2\lambda^4}\right) \rightarrow \left(\frac{k^4}{2\lambda^4}\right)^2 - 2\left(\frac{k^4}{2\lambda^4}\right) + \frac{\beta^2}{4\lambda^4U_T^2} = 0$$

$$\rightarrow \left(\frac{k^4}{2\lambda^4}\right) = 1 \pm \left(1-\frac{\beta^2}{4\lambda^4U_T^2}\right)^{1/2}$$

It is clear that the inclusion of β serves to stabilize the flow. (Recall that in the Eady analysis $\beta=0$ so we could not discern the role of β). With a meridional gradient in rotation present (through the β term) the thermal wind U_T must exceed the minimum $U_T = \frac{\beta}{2\lambda^2}$ in order for waves at any scale to become potentially unstable. As noted in the Eady problem, the flow is always stable for waves shorter than the critical wavelength

$$L_c = \sqrt{2}\pi\lambda^{-1}$$

The long wave stabilization associated with the β effect has to do with the rapid westward propagation of Rossby waves (recall $c-U_m = -\beta k^{-2}$ for Rossby waves). It can be shown that in QG dynamics, baroclinically unstable waves always propagate at a speed which lies between the max & min mean zonal wind speeds. Thus, in our two level model in the usual mid-latitude case where $U_1>U_3>0$, the real part of the phase speed c_r satisfies the the inequality $U_1>c_r>U_3$ for unstable waves. In a continuous atmosphere this would imply the existence of a level where $U=c_R$. Theoreticians call this the critical level; synopticians, the steering level. For long waves and weak basic state wind shear $c_R<U_3$ and so there is no steering level and unstable growth cannot occur.

Differentiating the condition $\delta=0$ w.r.t k and setting $\frac{\partial U_T}{\partial k}=0$, we find that the minimum value of U_T for which unstable waves may exist occurs when $k^2=\sqrt{2}\lambda^2$. This wavenumber corresponds to the wavenumber of maximum baroclinic instability. Wave-numbers of observed disturbences should be close to the wavenumber of max instability ($k^2=\sqrt{2}\lambda^2$), since if U_T were gradually raised from zero, the flow would first become unstable for perturbations of this wave length. Note that since $\lambda^2 = \frac{f_0^2}{\sigma(\Delta p)^2}$, the wavenumber of max instability decreases (wavelength increases) as the static stability increases. Again we see the stabilizing effect of vertical stratifications on the short wave end of the spectrum.

Those waves which do become unstable (with scales such that $k^2=\sqrt{2}\lambda^2$) initially grow and in the process remove energy from the mean thermal wind, thereby decreasing U_T and stabilizing the flow $\rightarrow$ we see baroclinic instability has a self limiting mechanism.

Under normal conditions

$$\lambda^2 \sim 2\times10^{-12}\ m^{-2}$$

So the most unstable wavelength is $\quad 2^{3/2}$ so $(2^{3/2})^{1/2} = 2^{3/4}$

$$k^2 = \sqrt{2}\lambda^2 = 2\sqrt{2}\times10^{-12}\ m^{-2}$$

$$\frac{2\pi}{L_x} = k = 2^{3/4}\times10^{-6}\ m^{-1}$$

$$L_x = (2\pi)(2^{3/4}\times10^{-6}\ m^{-1})^{-1}$$

$$L_x \sim 3736\ km$$

So the wavelength of max instability $\sim$ 4000km

With $\beta\sim10^{-12}\frac{1}{ms}$, the min thermal wind is given by

$$U_T \geq \frac{\beta}{2\lambda^2} = \frac{10^{-12}\ m^{-1}s^{-1}}{2\cdot2\times10^{-12}\ m^{-2}} = \frac{1}{4}\ m/s$$

$\rightarrow$ Holton says 4m/s.

Shears in the vertical of wind speed are commonly greater than this threshold. Therefore the observed behavior of mid-lat. system is consistent with the baroclinic instability mechanism.

• Vertical motions in baroclinic waves.

From QG dynamics (upon which the two-level model is based) we can apply the QG ω-eq. to assist in our understanding of the dynamics of baroclinic instability. It can be shown that the vertical field is related to other variables as follows.

① $-\omega_2' \propto -V_2' \frac{\partial T}{\partial y}$ → meridional perturbation flow advecting basic state temperature

② $-\omega_2' \propto -U_T \frac{\partial \zeta_2'}{\partial x}$ → thermal wind advecting perturbation vorticity.

The subscript "2" denotes the interface between the upper & lower levels in the model, generally 500mb. The $'$ denotes a perturbation.

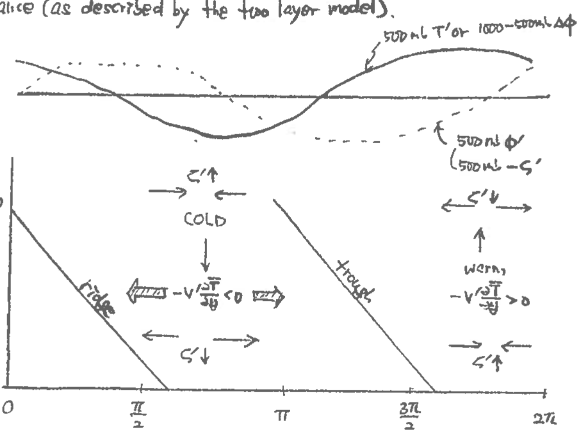
We may summarize as follows

① (cold / warm) air advection in the meridional direction by the perturbation meridional wind forces (sinking / rising) motion

② (negative / positive) vorticity advection in the zonal direction by the basic state thermal wind forces (sinking / rising) motion

The schematic below illustrates key features of a baroclinically unstable disturbance (as described by the two layer model).

Assume $U_T>0$ which is the typical case for mid-lat. baroclinic waves.



light arrows show direction of motion in secondary circulation
heavy arrows indicate Q vectors

At 500mb the thickness perturbation field lags (lies to to west) of the geopotential field by one quarter wavelength. The thickness and vertical motion field are in phase ⇒ rising motion ($\omega<0$) with falling heights, sinking motion ($\omega>0$) with rising heights. The temperature advection by the perturbation meridional wind is in phase with the 500mb thickness field. Thus advection of the basic state temperature field by the perturbation wind acts to intensify the perturbation thickness.

A divergence/convergence pattern contributes to

a (positive / negative) vorticity tendency at 250mb (trough / ridge)

and a (negative / positive) " " " 750mb (ridge / trough)

In all cases the vorticity tendencies due to the secondary circulation tend to increase the strength of the disturbance

The total vorticity change is due to both that due to vortex stretching from the divergent/convergent secondary circulation plus that due to advection. The advection of vorticity is relatively small at low levels but is dominant at upper levels. If no other processes influenced the vorticity field, the effect of differential vorticity advection would be to move the upper level trough & ridge pattern eastward more rapidly than the lower level pattern. This would destroy the westward tilt of the trough-ridge pattern. The maintenance of this tilt in the presence of differential vorticity advection is due to the concentration of vorticity by vortex stretching associated with divergent secondary circulation.

• Energetics

The total energy of the atmosphere is the sum of internal energy, gravitational potential energy, and kinetic energy (neglecting other smaller forms of energy such as chemical energy). However, it is not necessary to consider separately the variations of internal and gravitational potential energy because in a hydrostatic atmosphere these two forms of energy are proportional and may be combined into a single term called the total potential energy.

From $dE_I = \rho C_v T dz$ and $dE_p = \rho g z dz$, it can be shown

$$E_I = C_v\int_0^\infty \rho T dz \quad \text{and} \quad E_p = R\int_0^\infty \rho T dz$$

$p = \rho RT$

$E_p = \int_0^\infty \rho g z dz = -\int_{p_0}^0 z dp = \int_0^\infty p dz = R\int_0^\infty \rho$

And so $\frac{E_I}{C_v} = \frac{E_p}{R} \Rightarrow$ total P.E. $= E_I + E_p = \frac{C_p}{C_v}E_I = \frac{C_p}{R}E_p$

Total P.E. is not very useful because only a very small fraction of the total P.E. is available for conversion to K.E. Therefore we define the available P.E (APE).

APE ≡ the difference between the total P.E of a closed system and the minimum total P.E that could result from an adiabatic redistribution of mass.

Lorenz showed that the APE is given approximately by the volume integral over the entire atmosphere of the variance of potential temperature on isobaric sfc. Thus,

$$\overline{APE} \propto \left(\frac{1}{\text{volume}}\right)\int \frac{\overline{(\theta')^2}}{(\bar{\theta})^2}\, dV$$

∵ bar denotes average over isobaric sfc

total volume of atm

Observations indicate that for the atmosphere as a whole

input → $\frac{APE}{TPE} \sim 5\times10^{-3}$; $\frac{KE}{APE} \sim 10^{-1}$ ← output

only 0.5% of total P.E of atmosphere is available ; of the portion of APE only 10% is converted to KE.

From this standpoint the atmosphere has an efficiency of $\sim 5\times10^{-4}$ or 0.05%!

We can derive a set of perturbation energy eqs for our two level model. Let K' denote the perturbation KE summed from the values at 250 & 750mb. Let P' be the perturbation APE. We obtain the following perturbation energy eqs.

$$\frac{dK'}{dt} = -\left(\frac{2f_0}{\delta p}\right)\overline{\omega_2'\psi_T'}$$

← zonal average

$$\frac{dP'}{dt} = 4\lambda^2 U_T\, \overline{\psi_T'\frac{\partial \psi_m}{\partial x}} + \left(\frac{2f_0}{\delta p}\right)\overline{\omega_2'\psi_T'}$$

In words,

the rate of change in perturbation KE ∝ correlation between the perturbation thickness and vertical motion

the rate of change in perturbation APE ∝ −( ) + generation/dissipation of APE due to the correlation between the perturbation thickness and the meridional velocity at 500mb.

• Consider the conversion term $\left(\frac{2f_0}{\delta p}\right)\overline{\omega_2'\psi_T'}$

This term represents a conversion between potential & K.E.

warm air rising → $\psi_T'>0$, $+\omega_2'<0$ so $\overline{\omega_2'\psi_T'}<0$ → increase KE (decrease APE)

cold air sinking → $\psi_T'<0$, $\omega_2'>0$ so $\overline{\omega_2'\psi_T'}<0$

In this process the center of mass of the atmosphere is lowered and so P.E. is lost and appears as motions in the atmosphere.

We generally observe warm air rising east-NE of a sfc L and cold air sinking W-SW ⇒ hence a conversion from APE to KE of the cyclone.

Consider the generation term: $4\lambda^2 U_T\,\overline{\psi_T'\frac{\partial \psi_m}{\partial x}}$

To assist in understanding this term consider a particular sinusoid of a wave disturbance. Let

$\psi_m = A_m \cos k(x-ct)$; $\psi_T = A_T \cos k(x+x_0-ct)$ ← phase difference

↳ (barotropic part of perturbation ; ↳ (baroclinic part of perturbation

The phase angle kx_0 represents the difference in phase between the 500mb thickness (temperature) and geopotential fields. With this representation,

$$\overline{\psi_T'\frac{\partial\psi_m}{\partial x}} = \frac{1}{L}\int_0^L A_T A_m \cos k(x+x_0-ct)\sin k(x-ct)(-k)\,dx$$

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta)+i\sin(\alpha+\beta)$$

$$e^{i\alpha}e^{i\beta} = (\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta) = (\cos\alpha\cos\beta-\sin\alpha\sin\beta)+i(\cos\alpha\sin\beta+\sin\alpha\cos\beta)$$

So

$$\cos k(x+x_0-ct) = \cos k(x-ct)\cos kx_0 - \sin k(x-ct)\sin kx_0$$

and

$$\cos k(x+x_0-ct)\sin k(x-ct) = \underbrace{\cos kx_0\cos k(x-ct)\sin k(x-ct)}_{\text{Integrate this over one period and we get zero.}} - \sin kx_0\left[\sin k(x-ct)\right]^2$$

So,

$$\overline{\psi_T'\frac{\partial\psi_m}{\partial x}} = +\frac{k}{L}\sin kx_0\underbrace{\int_0^L A_m A_T \sin^2 k(x-ct)\,dx}_{\text{this is } 1/2}$$

$$= A_m A_T k\sin kx_0/2$$

With $U_T>0$ generally for mid-latitudes $\overline{\psi_T'\frac{\partial\psi_m}{\partial x}}$ must be positive for the generation of perturbation APE (call it EAPE). The sinusoid $\sin kx_0>0$ for $0<kx_0<\pi$ with a max at $kx_0=\frac{\pi}{2}$. Thus, when the thickness (temperature) field at 500mb lags the geopotential at 500mb by 90° (¼ wavelength) we have the maximum generation of EAPE.

This is what we find in the observations. When the temperature waves lags the geopotential by ¼ cycle, the northward advection of warm air by geostrophic wind east of the 500mb trough and the southward advection of cold air west of the 500mb trough are both maximized. As a result cold advection is strong below the 250mb trough and warm advection is strong beneath the 250mb ridge → the upper level disturbance will intensify. Also note that the above lag implies that the trough and ridge axes tilt westward with height. (for amplifying mid-lat synoptic systems)

We've already noted that the $\overline{\omega'\psi_T'}$ term is positive both east & west of the 500mb trough.
warm air rising / cold air sinking

Thus for QG perturbations, a westward tilt of the perturbation with height implies both that

① the horizontal temp. advection will increase the EAPE
and
② the vertical circulation will convert EAPE to EKE.

An eastward tilt would change the direction of the energy flow.

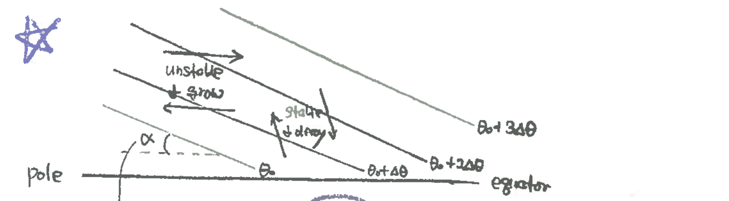
If we add the EAPE & EKE eqs to get the total energy of the disturbance, we find

$$\frac{d}{dt}(P'+K') = 4\lambda^2 U_T\,\overline{\psi_T'\frac{\partial\psi_m}{\partial x}} \qquad \lambda^2=\frac{f_0^2}{\sigma(\Delta p)^2}$$

Thus, provided the correlation between the meridional velocity and the temperature is positive and $U_T>0$, the total energy of the perturbation will grow. The vertical circulation merely converts EAPE ↔ EKE without affecting the total energy of the perturbation. The rate of increase/decrease in the total energy of the perturbation depends upon the magnitude of the basic state thermal wind U_T. This in turn is proportional to the meridional temperature gradient. Since the generation of EAPE requires a systematic poleward transport of warm air and an equatorward transport of cold air, it is clear that baroclinic eddies tend to reduce the meridional temperature gradient and hence the APE of the mean flow.

We can see qualitatively that if parcels move poleward (equatorward) and upward (downward) with slopes less than the slope of the zonal mean potential temperature sfc, they will become warmer (colder) than their surroundings. For such parcels the correlations $\overline{v'\theta'}$ and $\overline{\omega'\theta'}$ will be positive as required for baroclinically unstable disturbances. For parcels that have slopes greater than that of the zonal mean θ slope, the correlation will be negative. Such parcels must convert EKE to EAPE which in turn is converted into ZAPE. Therefore, in order that eddies be able to extract PE from the mean flow, the perturbation parcel trajectories must have slopes less than the slopes of the θ sfc. Further, a permanent rearrangement of air must take place for there to be a net heat transfer.

These ideas are illustrated below.



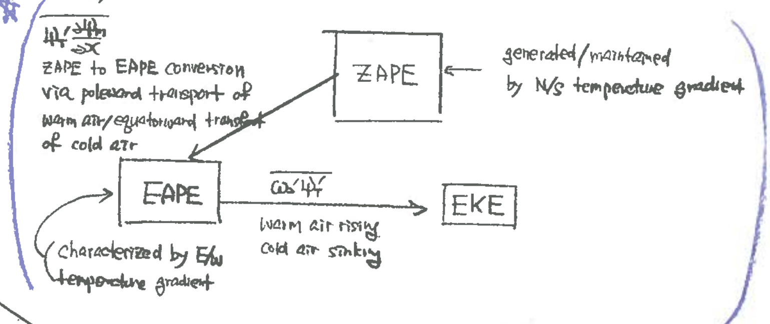
This angle α is called the Eady angle. If the parcel rises along a slanted trajectory with a slope angle $\beta\in(0,\alpha)$ then "slant-wise convection" will occur and the parcel will continue to rise. ZAPE is converted to EAPE.

Since we have seen that poleward moving air must rise & equatorward moving air must sink, it is clear that the rate of energy generation can be greater for an atmosphere in which the meridional slope of the θ sfcs is large (large $\frac{\partial\overline{\theta}}{\partial y}$)

We can also see why baroclinic instability has a short wave cutoff. The intensity of the vertical circulation must increase as the wavelength decreases in order to keep temperature changes hydrostatic & vorticity changes geostrophic. Thus the slopes of parcel trajectories increase as the wavelength of the disturbance decrease. For some critical wavelength the trajectory slope exceeds the Eady angle and development via the mechanism of baroclinic instability stops.

→ Baroclinic instability prefers perturbations of an intermediate range of scales (~ synoptic scale). This is unlike convective instability where the most rapid growth occurs on the smallest scales.

The energy flow in QG disturbances is summarized below.



* Kelvin's theorem

OCP5253 (Geophysical Fluid Dynamics): Ahlquist (60 minutes) **

• Kelvin's theorem says that circulation around a curve C is conserved following fluid motion if the fluid is barotropic on C and if friction vanishes on C. $\frac{dC_a}{dt}=0$

a) Derive Kelvin's theorem from the vector equation of motion.

b) Prove that circulation around a vortex tube is independent of where the curve C circles the vortex tube. ← use divergence + Stokes theorems!

c) Prove that vortex tubes move with the fluid when Kelvin's theorem applies.

d) Explain how the conservation of potential vorticity is really Kelvin's theorem for a specially chosen contour.

Solution → look at Holton's sections 4.1 + 4.3

note

- Circulation and vorticity are two primary measures of rotation in a fluid.
  - Circulation – scalar – macroscopic (integral) measure of rotation for a finite area of the fluid
  - Vorticity – vector – microscopic measure of rotation at any point in fluid.
- The circulation C about a closed contour in a fluid is defined as the line integral about the contour of the component of the velocity vector that is locally tangent to the contour:

$$\boxed{C \equiv \oint \underset{\sim}{U}\cdot d\underset{\sim}{l}} = \oint |\underset{\sim}{U}|\cos\phi\, dl \qquad (1)$$

By convention the circulation is taken to be positive if $C>0$ for counter-clockwise integration around the contour.

(a) The circulation theorem is obtained by taking the line integral of Newton's second law for a closed chain of fluid particles. In the absolute coordinate system and neglecting viscous (frictional) forces, Newton's second law may be written as;

$$\frac{d_a \underset{\sim}{U}_a}{dt} = -\frac{1}{\rho}\nabla p - \nabla\Phi \qquad (2)$$

where $\frac{d_a}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{U}_a\cdot\nabla$ is the total derivative w.r.t time following the motion

$-\frac{1}{\rho}\nabla p$ is the pressure gradient force

$-\nabla\Phi = \underset{\sim}{g}$ is the gravitational acceleration. We have represented $\underset{\sim}{g}$ as the gradient of a scalar, Φ. the geopotential

We now take the line integral of Newton's second law for a closed chain of fluid particles

$$\oint \frac{d_a \underset{\sim}{U}_a}{dt}\cdot d\underset{\sim}{l} = -\oint \frac{\nabla p\cdot d\underset{\sim}{l}}{\rho} - \oint \nabla\Phi\cdot d\underset{\sim}{l} \qquad (3)$$

For the moment only consider the LHS of (3). The integrand may be rewritten as

$$\frac{d_a \underset{\sim}{U}_a}{dt}\cdot d\underset{\sim}{l} = \frac{d_a}{dt}(\underset{\sim}{U}_a\cdot d\underset{\sim}{l}) - \underset{\sim}{U}_a\cdot\frac{d_a}{dt}(d\underset{\sim}{l})$$

Note that for a scalar (such as $\underset{\sim}{U}_a\cdot d\underset{\sim}{l}$) the rate of change following the motion does not depend on the reference system. For a vector, the rate of change following the motion does depend on the reference frame, but not for a scalar. Therefore

$$\frac{d_a}{dt}(\underset{\sim}{U}_a\cdot d\underset{\sim}{l}) = \frac{d}{dt}(\underset{\sim}{U}_a\cdot d\underset{\sim}{l})$$

Since $\underset{\sim}{l}$ is a position vector, $\frac{d_a \underset{\sim}{l}}{dt} = \underset{\sim}{U}_a$ so $\frac{d_a}{dt}(d\underset{\sim}{l}) = d\underset{\sim}{U}_a$

Thus, $\frac{d_a \underset{\sim}{U}_a}{dt}\cdot d\underset{\sim}{l} = \frac{d}{dt}(\underset{\sim}{U}_a\cdot d\underset{\sim}{l}) - \underset{\sim}{U}_a\cdot d\underset{\sim}{U}_a$

Substitute this result into (3)

$$\oint \frac{d}{dt}(\underset{\sim}{U}_a\cdot d\underset{\sim}{l}) - \oint \underset{\sim}{U}_a\cdot d\underset{\sim}{U}_a = -\oint\frac{\nabla p\cdot d\underset{\sim}{l}}{\rho} - \oint\nabla\Phi\cdot d\underset{\sim}{l} \qquad (4)$$

Recall that the line integral about a closed loop of a perfect differential is zero

Therefore $\oint \nabla\Phi\cdot d\underset{\sim}{l} = \oint d\Phi = 0$ ✓

Also note that

$$\oint \underset{\sim}{U}_a\cdot d\underset{\sim}{U}_a = \frac{1}{2}\oint d(\underset{\sim}{U}_a\cdot\underset{\sim}{U}_a) = 0 \;✓$$

Thus the above integral eq. reduces to

$$\frac{d}{dt}\oint \underset{\sim}{U}_a\cdot d\underset{\sim}{l} = -\oint\frac{\nabla p\cdot d\underset{\sim}{l}}{\rho} \qquad (5)$$

But by definition, $\oint \underset{\sim}{U}_a\cdot d\underset{\sim}{l} = C_a$, the circulation (absolute because we use $\underset{\sim}{U}_a$)

Also note that we can write $\nabla p\cdot d\underset{\sim}{l} = dp$

With these changes we obtain the (absolute) circulation theorem:

$$\boxed{\frac{dC_a}{dt} = -\oint\frac{dp}{\rho}} \qquad (6)$$

The term $-\oint\frac{dp}{\rho}$ is called the solenoidal term. For a barotropic fluid the density ρ is a function of only pressure. Thus we can write $\frac{dp}{\rho}$ as $d(F(p))$ where F is a function of pressure which involved the density ρ. So, in a barotropic fluid we can rewrite $\oint\frac{dp}{\rho} = \oint d(F(p)) = 0$ ← perfect differential

We are left with Kelvin's Circulation theorem

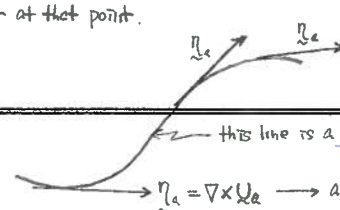
$$\frac{dC_a}{dt} = 0$$

In a barotropic fluid, the absolute circulation is conserved following the motion. This result is the fluid analog of angular momentum conservation in solid body mechanics.

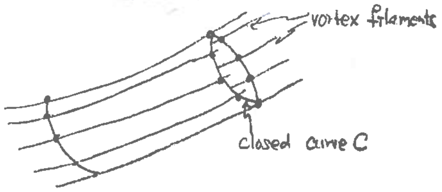
(b) Pedlosky, Section 2.1

First, what do we mean by vortex tube?

We first start with a definition of a vortex line or vortex filament. A vortex filament is a line in the fluid which at each point is parallel to the vorticity vector at that point.



A vortex tube is formed by the surface consisting of the vortex filaments which pass through a closed curve C



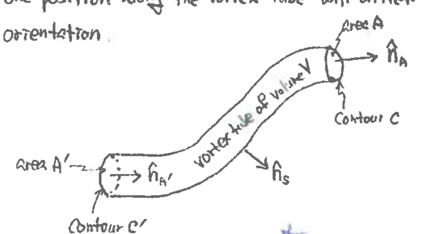
Since the filaments change their orientation from one position to another, the boundary curve C′ of the tube at another position along the vortex tube will differ from C in both length and orientation. However, by definition, at all points along the surface of the tube, there can be no component of vorticity which penetrates the tube surface. Further since the vorticity vector $\underset{\sim}{\eta}_a$ is the curl of the velocity field $\underset{\sim}{V}_a$, the divergence of $\underset{\sim}{\eta}_a$ is zero. That is, the vorticity vector $\underset{\sim}{\eta}_a$ is divergence free. It is a nondivergent vector field.

$$\nabla\cdot\underset{\sim}{\eta}_a = \nabla\cdot(\nabla\times\underset{\sim}{V}_a) = 0$$

Thus, since $\underset{\sim}{\eta}_a$ is divergence free, it follows that for arbitrary volumes V

$$\iiint_V (\nabla\cdot\underset{\sim}{\eta}_a)\,dV = 0 \qquad (\text{since } \nabla\cdot\underset{\sim}{\eta}_a = 0)$$

Now consider the volume of integration consisting of any finite segment of the vortex tube sliced at C and C′ by plane surfaces with areas A and A′, respectively. As noted before, the boundary curve C′ of the tube at one position along the vortex tube will differ from C in both length and orientation.



From the divergence theorem

$$\boxed{\iiint_V (\nabla\cdot\underset{\sim}{\eta}_a)\,dV = \iint_A \underset{\sim}{\eta}_a\cdot\hat{n}\,dA}$$

where $\hat{n}$ is the outward normal at each point on the volume's bounding surface. Since $\underset{\sim}{\eta}_a\cdot\hat{n}$ is identically zero on the surface formed by the vortex filaments the only contribution to a volume integral of $\nabla\cdot\underset{\sim}{\eta}_a$ over the volume V (indicated above) comes from the ends of the tube.

That is

$$\iiint_V (\nabla\cdot\underset{\sim}{\eta}_a)\,dV = \underbrace{\iint_{A_s}\cancelto{0}{(\underset{\sim}{\eta}_a\cdot\hat{n}_s)}\,dA_s}_{\text{This is zero since } \underset{\sim}{\eta}_a\perp\hat{n}_s \text{ at all points}} + \underbrace{\iint_{A'}(\underset{\sim}{\eta}_a\cdot(-\hat{n}_{A'}))\,dA'}_{\text{must use } -\hat{n}_{A'}\text{, since the normal must point out of the volume}} + \iint_A(\underset{\sim}{\eta}_a\cdot\hat{n}_A)\,dA$$

But we have already noted that the vorticity vector $\underset{\sim}{\eta}_a$ is nondivergent. This means the volume integral is identically zero. We are left with

$$0 = -\iint_{A'}(\underset{\sim}{\eta}_a\cdot\hat{n}_{A'})\,dA' + \iint_A(\underset{\sim}{\eta}_a\cdot\hat{n}_A)\,dA$$

or

$$\iint_{A'}(\underset{\sim}{\eta}_a\cdot\hat{n}_{A'})\,dA' = \iint_A(\underset{\sim}{\eta}_a\cdot\hat{n}_A)\,dA$$

By definition the circulation Γ_c around a contour C which bounds an area A is given as

$$\Gamma_c = \oint_c \underset{\sim}{V}_a\cdot d\underset{\sim}{l} \overset{\text{apply Stoke's theorem}}{=} \iint_A(\nabla\times\underset{\sim}{V}_a)\cdot\hat{n}_A\,dA$$

or

$$\Gamma_c = \oint_c \underset{\sim}{V}_a\cdot d\underset{\sim}{l} = \iint_A(\underset{\sim}{\omega}_a\cdot\underset{\sim}{\eta}_a)\,dA$$

From the above result we find $\Gamma_{c'} = \Gamma_c$. That is, the circulation around a vortex tube is independent of where the contour C circles the vortex tube. The strength (circulation) of a vortex tube is constant along the length of the tube. An important consequence of this result is that absolute vortex tubes and the filaments of which they are composed can not end in the interior of the fluid

(c) Pedlosky section 2.3

We start from the circulation theorem in an absolute reference frame. Let Γ_a denote the absolute circulation around a closed contour C. The time rate change in Γ_a following the motion is given by

$$\frac{d\Gamma_a}{dt} = -\oint_c \frac{\nabla p}{\rho}\cdot d\underset{\sim}{l} + \oint_c \underbrace{\frac{\underset{\sim}{F}}{\rho}}_{\text{frictional forces}}\cdot d\underset{\sim}{l}$$

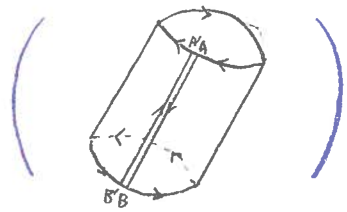
Now if ① the fluid is barotropic on C $\Rightarrow \oint \frac{\nabla p}{\rho}\cdot d\underset{\sim}{l} = 0$ since $\frac{\nabla p}{\rho}$ becomes a function of p alone and $\oint$(perfect diff)
and
if ② the friction force vanishes on C $\Rightarrow \underset{\sim}{F} = 0$

then the absolute circulation Γ_a is conserved following the motion, i.e.

$$\boxed{\frac{d\Gamma_a}{dt} = 0}$$

This is Kelvin's theorem. We see that the conservation of Γ_a means that as the closed contour C moves and deforms with the fluid, it always forms the boundary of a tube of fixed absolute vortex strength. If the cross section of a tube decreases, the intensity of the tube strength (vorticity $\underset{\sim}{\eta}_a$) in the tube must increase in inverse proportion to the tube cross sectional area.

Consider now the absolute vorticity tube shown below



The tube has been wrapped initially with a cylindrical "blanket" whose four corners are A, A′, B, B′. Since at this initial instant this surface wraps a vortex tube, there is no flux of vorticity through the surface and the circulation about the circuit $A \to B \to B' \to A' \to A$ must also vanish. Let this blanket now move with the fluid. If Kelvin's theorem applies, the circulation about the circuit must remain zero. That is, no absolute vorticity filaments can penetrate the blanket. In the limit where A′ coalesces with A while B′ coalesces with B, it follows that the surface of the absolute vorticity tube remains coincident with the "blanket" of material elements. In other words, absolute vortex tubes move with the fluid when Kelvin's theorem applies.

In the limit of a vortex tube of vanishingly small section we find the following result. ⇒ When Kelvin's theorem is valid, individual filaments of absolute vorticity remain material lines. That is, a line which is once a filament of absolute vorticity is always a filament so that absolute vorticity filaments move with the fluid.

(d)

We start with the circulation theorem

$$\frac{dC_a}{dt} = \frac{d}{dt}\oint \underset{\sim}{V}_a\cdot d\underset{\sim}{l} = -\oint \frac{\nabla p\cdot d\underset{\sim}{l}}{\rho} \qquad (1)$$

Recall Stoke's theorem,

$$\boxed{\underbrace{\iint_A(\nabla\times\underset{\sim}{a})\cdot\hat{n}\,dA}_{\text{sum of the component of the curl of } \underset{\sim}{a} \text{ normal to the surface enclosed by contour C}} = \underbrace{\oint_c \underset{\sim}{a}\cdot d\underset{\sim}{l}}_{\text{line integral of } \underset{\sim}{a} \text{ around a closed contour C}}} \qquad (2)$$

With regards to circulation, Stoke's theorem is very revealing. Via Stokes theorem we see that the circulation around a closed contour equals the sum of all the vorticity components normal to the closed surface. That is,

$$C \equiv \oint \underset{\sim}{U} \cdot d\underset{\sim}{l} = \iint (\nabla \times \underset{\sim}{U}) \cdot \hat{n}\, dA$$

or

$$C = \iint \underset{\sim}{\eta} \cdot \hat{n}\, dA \qquad \because \underset{\sim}{\eta} = \nabla \times \underset{\sim}{U} \text{ is the 3D vorticity vector}$$

This supports a statement we made earlier. Namely, circulation is a macroscopic measure of fluid rotation whereas vorticity is a microscopic (at a point) measure of rotation.

By using Stoke's theorem we can rewrite the circulation theorem as

$$\frac{d}{dt}\iint_A (\nabla \times \underset{\sim}{U}_a) \cdot \hat{n}\, dA = -\iint_A \nabla \times \left(\frac{\nabla p}{\rho}\right) \cdot \hat{n}\, dA \qquad (3)$$

using the vector identity

$$\nabla \times \alpha \underset{\sim}{A} = \nabla \alpha \times \underset{\sim}{A} + \alpha \nabla \times \underset{\sim}{A}$$

we rewrite

$$\nabla \times \frac{\nabla p}{\rho} = \nabla\left(\frac{1}{\rho}\right) \times \nabla p + \frac{1}{\rho} \nabla \times \nabla p$$

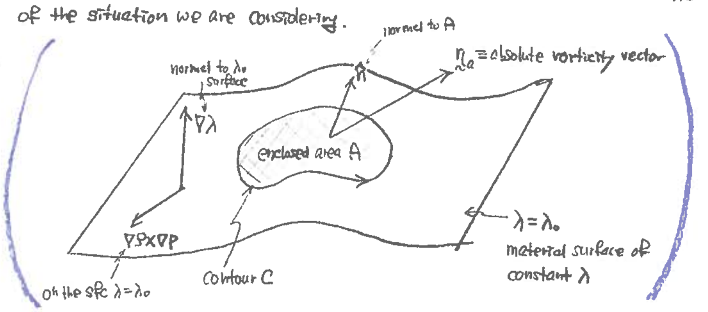
$$= -\frac{1}{\rho^2}(\nabla \rho \times \nabla p) + 0 \longleftarrow \text{the curl of the gradient is zero.}$$

We further define the absolute vorticity $\underset{\sim}{\eta}_a = \nabla \times \underset{\sim}{U}_a$

Using all of this we rewrite the circulation theorem above as

$$\boxed{\frac{d}{dt}\iint_A \underset{\sim}{\eta}_a \cdot \hat{n}\, dA = +\iint_A \frac{\nabla \rho \times \nabla p}{\rho^2} \cdot \hat{n}\, dA} \quad (4) \quad \because (\nabla \rho \times \nabla p \text{ is the baroclinic vector})$$

The area A is an area on any material surface enclosed by the contour C. $\hat{n}$ is a unit vector normal to the surface. Suppose that the contour C which encloses A is chosen to lie initially on a surface of constant λ (λ is some scalar). Lets take the particular constant λ surface to be λ_0. On the below we draw a schematic of the situation we are considering.



Since λ is conserved by each fluid element on the sfc $\lambda = \lambda_0$ we call the surface $\lambda = \lambda_0$ a material surface. The closed contour C on a material surface is called a material line. A material line remains on its material surface.

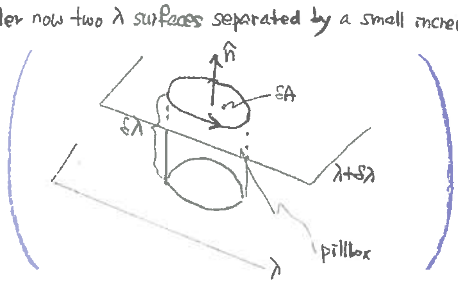
Lets assume λ is a function of p and ρ alone. The vector $\nabla \rho \times \nabla p$ lies on the surface of constant λ. Naturally $\nabla \lambda$ is normal to the constant λ surface.

If we apply Kelvin's theorem, the RHS of (1) vanishes (since $(\nabla \rho \times \nabla p) \perp \hat{n}$). The same is true in (4) under the assumption that Kelvin's theorem applies. In this case we have

$$\frac{d}{dt}\iint_A \underset{\sim}{\eta}_a \cdot \hat{n}\, dA = 0 \qquad (5)$$

This is true for all time t. Anticipating later results the above eq shows that $\iint_A \underset{\sim}{\eta}_a \cdot \hat{n}\, dA$ is conserved following the motion.

Consider now two λ surfaces separated by a small increment $\delta\lambda$



The separation between the surfaces is $\delta\lambda$ where $\delta\lambda |\nabla \lambda| = \Delta\lambda$

The mass of the pillbox is

$$\delta m = \rho\, \delta A\, \delta\lambda \quad \text{or} \quad \delta m = \rho\, \delta A \frac{\Delta\lambda}{|\nabla\lambda|}$$

Since the boundaries (upper + lower) of the box are material surfaces and we assume the sides of the pillbox are similarly material surfaces, conservation of mass applies $\Rightarrow \delta m$ is constant following the motion of the pillbox. We now consider the pillbox to be so small that

$$\iint_A \underset{\sim}{\eta}_a \cdot \hat{n}\, dA \simeq (\underset{\sim}{\eta}_a \cdot \hat{n})\, \delta A$$

We introduce this approximation into (5) to obtain a specialized variant of Kelvin's theorem

$$\frac{d}{dt}\left((\underset{\sim}{\eta}_a \cdot \hat{n})\, \delta A\right) = 0 \qquad (6)$$

But we also have

$$\frac{d}{dt}(\delta m) = 0$$

With the mass element of δm constant we can solve for the area element δA using the relation $\delta m = \rho\, \delta A \frac{\Delta\lambda}{|\nabla\lambda|} \longrightarrow \delta A = \frac{|\nabla\lambda|}{\Delta\lambda}\frac{\delta m}{\rho}$

We use this in (6) to obtain

$$\frac{d}{dt}\left[(\underset{\sim}{\eta}_a \cdot \hat{n}) \frac{|\nabla\lambda|}{\Delta\lambda}\frac{\delta m}{\rho}\right] = 0$$

But $\nabla\lambda = |\nabla\lambda|\hat{n}$. Furthermore δm and $\Delta\lambda$ are constant following the motion so they may be removed. We are left with

$$\frac{d}{dt}\left[\frac{\underset{\sim}{\eta}_a \cdot \nabla\lambda}{\rho}\right] = 0$$

That is, the quantity $Q = \frac{\underset{\sim}{\eta}_a \cdot \nabla\lambda}{\rho}$ is conserved following the motion.

If we now let λ be potential temperature,

$$Q = \frac{\underset{\sim}{\eta}_a \cdot \nabla\theta}{\rho}$$

We call Q as defined above potential vorticity and the conservation of potential vorticity following the motion is referred to as Ertel's theorem. We see that conservation of potential vorticity is really Kelvin's theorem for a very special contour $C \longrightarrow$ the contour C is a potential temperature sfc

---

(d) again based on Holton.

Start with the definition of potential temperature θ.

$$\checkmark \quad \theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

From the equation of state $p = \rho R T \Longrightarrow T = \frac{p}{\rho R}$ so

$$\theta = \frac{p}{\rho R}\left(\frac{p_0}{p}\right)^{R/c_p}$$

Solve for ρ

$$\rho = (R\theta)^{-1} p^{(1 - R/c_p)} p_0^{R/c_p} \qquad 1 - \frac{R}{c_p} = \frac{c_p - R}{c_p} = \frac{c_v + R - R}{c_p} = \frac{c_v}{c_p}$$

or

$$\rho = (R\theta)^{-1} p^{c_v/c_p} p_0^{R/c_p}$$

This result shows that on an isentropic surface (constant θ) the density ρ is a function of pressure alone. In this case the circulation theorem

$$\frac{dC_a}{dt} = \frac{d}{dt}\oint \underset{\sim}{U}_a \cdot d\underset{\sim}{l} = -\oint \frac{\nabla p}{\rho} \cdot d\underset{\sim}{l}$$

on an isentropic sfc $\rho = \rho(p)$ so we have the line integral of a perfect differential around a closed contour. This is identically zero.

collapses to

$$\frac{dC_a}{dt} = \frac{d}{dt}\left(\oint \underset{\sim}{U}_a \cdot d\underset{\sim}{l}\right) = 0$$

Or via $\boxed{\text{Stoke's theorem}}$

$$\frac{d}{dt}\iint_A (\nabla \times \underset{\sim}{U}_a) \cdot \hat{n}\, dA = 0 \qquad \because \nabla \times \underset{\sim}{U}_a \equiv \underset{\sim}{\eta}_a \rightarrow \text{absolute vorticity vector}$$

$$\frac{d}{dt}\iint_A (\underset{\sim}{\eta}_a \cdot \hat{n})\, dA = 0$$

Thus, for adiabatic flow the circulation computed for a closed chain of fluid parcels on a constant θ surface reduces to the same form as in a barotropic fluid. In this situation the circulation satisfies Kelvin's theorem even though we have not assumed the fluid is barotropic.

The result $\frac{d}{dt}\iint_A (\underset{\sim}{\eta}_a \cdot \hat{n})\, dA = 0$

Implies $\iint_A (\underset{\sim}{\eta}_a \cdot \hat{n})\, dA = \text{constant}.$

That is, the component of the absolute vorticity vector normal to the isentropic sfc summed over the area A enclosed by the contour is constant. If the area A is very small then we may approximately say

$$\iint_A (\underset{\sim}{\eta}_a \cdot \hat{n})\, dA \simeq (\underset{\sim}{\eta}_a \cdot \hat{n})\,\delta A = \text{constant}.$$

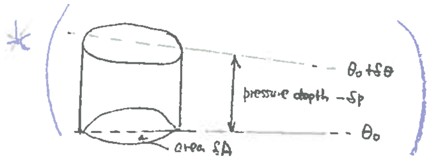
where δA is the small area enclosed by the contour on the θ sfc. Then

$$\eta_\theta\, \delta A = \text{constant}.$$

Suppose the small parcel we are considering is confined between θ surfaces θ_0 and $\theta_0 + \delta\theta$ where $|\delta\theta|$ is very, very small. Assume the pressure difference between these θ surfaces to be $-\delta p$

($-$ sign is included since p generally $\downarrow$ as $\theta\uparrow$)

We have the situation shown below



The mass of the parcel is conserved following the motion, where the mass δM is given by $\delta M = \left(-\frac{\delta p}{g}\right)\delta A$

We solve this for the area

$$\delta A = (-g)\frac{\delta M}{\delta p} = (-g)\left(\frac{\delta\theta}{\delta p}\right)\left(\frac{\delta M}{\delta\theta}\right) \quad \because \delta\theta \text{ is constant.}$$

We replace δA in our expression for η_θ on the above.

$$\eta_\theta\, \delta A = \text{constant}$$

becomes

$$\eta_\theta\, g\left(-\frac{\delta\theta}{\delta p}\right)\left(\frac{\delta M}{\delta\theta}\right) = \text{const.}$$

or

$$\eta_\theta\, g\left(-\frac{\delta\theta}{\delta p}\right) = \text{constant} \leftarrow \frac{\delta M}{\delta\theta} \text{ is constant so absorb it into the RHS.}$$

In the limit as $\delta p \to 0$

$$\boxed{P = \eta_\theta\left(-g\frac{\partial\theta}{\partial p}\right) = \text{constant.}}$$

The quantity P is the isentropic coordinate form of Ertel's potential vorticity. The above result shows that P is conserved following the motion in adiabatic, frictionless flow. Potential vorticity is always in some sense a measure of the ratio of the absolute vorticity to the effective depth of the vortex.

This discussion has shown that the idea of conservation of potential vorticity is really just a form of Kelvin's theorem when the contour is chosen to be on an isentropic surface.

* Circulation theorem

OCP5253 (Geophysical fluid dynamics): Ahlquist (45 minutes) *

• The circulation theorem for frictionless flow in an inertial reference frame is

$$\frac{d}{dt}\oint_\Gamma \mathbf{V}_a \cdot d\mathbf{l} = -\oint_\Gamma \frac{1}{\rho}\nabla p \cdot d\mathbf{l} \qquad (1)$$

where the closed loop Γ for the line integral moves with the fluid.

(a) Show that the circulation theorem can also be written as

$$\frac{d}{dt}\iint_A \underset{\sim}{\zeta}_a \cdot \mathbf{n}dA = \iint_A \frac{\nabla\rho \times \nabla p}{\rho^2}\cdot \mathbf{n}dA \qquad (2)$$

where A is any area whose boundary is Γ and where $\underset{\sim}{\zeta}_a$ is the absolute vorticity vector. (Do not prove Gauss' theorem (the divergence theorem), but you do need to write it.)

(b) We shall apply the circulation theorem to the atmosphere. Let $\theta = \rho^{-1}p^{1/\gamma}\left(p_0^{R/c_p}/R\right)$ represent potential temperature, and assume that the flow conserves potential temperature. If loop Γ and area A lie in a potential temperature surface, solve for ρ as a function of p and θ and show that

$$\iint_A \frac{\nabla\rho\times\nabla p}{\rho^2}\cdot\mathbf{n}dA = 0$$

(c) If A is just dA, then $\iint \underset{\sim}{\zeta}_a \cdot \mathbf{n}dA = \underset{\sim}{\zeta}_a \cdot \mathbf{n}dA$, so with the results from (a) and (b), $d/dt(\zeta_a \cdot \mathbf{n}dA) = 0$. Consider two nearby θ surfaces separated by distance δl. Using the fact that the mass in the cylinder defined by Γ and δl is conserved, show that $\mathbf{n}dA \propto \frac{1}{\rho}\nabla\theta$, so $\frac{d}{dt}\left(\frac{\zeta_a\cdot\nabla\theta}{\rho}\right) = 0$

(a)

via Stokes theorem

$$\oint_\Gamma \underset{\sim}{V}_a \cdot d\underset{\sim}{l} = \iint_A \nabla\times\underset{\sim}{V}_a \cdot \hat{n}\, dA$$

But by definition

$$\underset{\sim}{\zeta}_a = \nabla\times\underset{\sim}{V}$$

↳ 3D vorticity vector

So $\oint_\Gamma \underset{\sim}{V}\cdot d\underset{\sim}{l} = \iint_A \underset{\sim}{\zeta}_a \cdot \hat{n}dA$

Next apply Stokes theorem to the RHS of (1). Doing so yields

$$-\oint \frac{1}{\rho}\nabla p \cdot d\underset{\sim}{l} = -\iint_A \nabla\times\left(\frac{\nabla p}{\rho}\right)\cdot\hat{n}dA$$

But

$$\nabla\times\left(\frac{\nabla p}{\rho}\right) = -\frac{1}{\rho^2}\nabla\rho\times\nabla p + \frac{1}{\rho}\underbrace{\nabla\times\nabla p}_{\text{the curl of a gradient is zero}}$$

$$= -\frac{\nabla\rho\times\nabla p}{\rho^2}$$

So

$$-\oint_\Gamma \frac{\nabla p}{\rho}\cdot d\underset{\sim}{l} = +\iint_A \frac{\nabla\rho\times\nabla p}{\rho^2}\cdot\hat{n}dA$$

Putting the pieces together we see that we may rewrite (1) as

$$\frac{d}{dt}\iint_A \underset{\sim}{\zeta}_a\cdot\hat{n}dA = +\iint_A \frac{\nabla\rho\times\nabla p}{\rho^2}\cdot\hat{n}dA \qquad (2)$$

(b)

From the given definition of θ, the density ρ is given as

$$\rho = p^{1/\gamma}\left(\frac{P_0^{R/c_p}}{R\theta}\right)$$

If θ is constant (as it is on an isentropic surface), then

$$\nabla\rho = \left(\frac{P_0^{R/c_p}}{R\theta}\right)\left(\frac{1}{\gamma}p^{1/\gamma-1}\nabla p\right)$$

then

$$\nabla\rho\times\nabla p = \underbrace{\left(\frac{P_0^{R/c_p}}{R\theta}\right)\left(\frac{1}{\gamma}p^{\frac{1}{\gamma}-1}\right)\nabla p}\times\nabla p$$

This vector is identical to ∇p in orientation. It only differs in magnitude. Generally with atmospheric values of θ and p, this vector and ∇p will be parallel. The cross product of parallel or anti-parallel vectors is zero.

Thus $\nabla\rho\times\nabla p = 0$.

and $\iint_A \frac{\nabla\rho\times\nabla p}{\rho^2}\cdot\hat{n}dA = 0$,

when contour Γ and the area A it encloses lie on a constant θ surface.

Note that this result implies that circulation around a closed contour Γ on a θ sfc is constant (conserved)

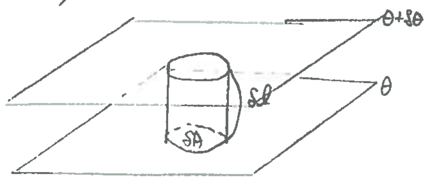
(c)

We want to find an alternative expression for δA. We do so by considering the mass contained in a tiny cylindrical pillbox between two surfaces of constant θ. The separation distance between the surfaces is δl. Note that δl is related to the value of θ on the neighboring surface by

$$\delta l\,|\nabla\theta| = \delta\theta \quad \text{or equally} \quad |\nabla\theta| = \frac{\delta\theta}{\delta l}$$

In writing this equality we have made use of the definition of $\nabla\theta$,

$$|\nabla\theta| = \frac{\delta\theta}{\delta l}$$



The mass of the pillbox is

$$\delta M = \rho\,\delta A\,\delta l$$

or

$$\delta M = \rho\,\delta A\left(\frac{\delta\theta}{|\nabla\theta|}\right)$$

Since the θ surfaces are material surfaces in this problem δM, the mass, is conserved following the motion of the pillbox. That is

$$\frac{d}{dt}(\delta M) = 0 \quad \Rightarrow \delta M = \text{const}$$

We also have for the same pillbox

$$\frac{d}{dt}(\underset{\sim}{\zeta}_a\cdot\hat{n}\,\delta A) = 0 \quad \Rightarrow \underset{\sim}{\zeta}_a\cdot\hat{n}\,\delta A = \text{const.}$$

We can express δA in terms of δM

$$\delta A = \frac{\delta M}{\rho}\frac{|\nabla\theta|}{\delta\theta}\cdot\text{constant}$$

Therefore

$$\underset{\sim}{\zeta}_a\cdot\hat{n}\,\frac{\delta M}{\rho}\frac{|\nabla\theta|}{\delta\theta}\cdot\text{constant} = \text{const.}$$

Since δM and $\delta\theta$ are constant following the motion and

$$\hat{n}\,|\nabla\theta| = \nabla\theta$$

We have the result

$$\frac{\underset{\sim}{\zeta}_a\cdot\nabla\theta}{\rho} = \text{const.}$$

or

$$\frac{d}{dt}\Bigg(\underbrace{\frac{\underset{\sim}{\zeta}_a\cdot\nabla\theta}{\rho}}_{\text{Potential vorticity}}\Bigg) = 0$$

* Rossby wave reflection

OCP5253 (Rossby wave reflection): Ahlquist (60 minutes) **

- Consider the Rossby wave dispersion relation

$$\underset{\text{frequency}}{\sigma} = -\frac{\beta k}{k^2 + l^2 + F}$$

where k and l are the x and y wavenumbers (2π/wavelength) and where β and F are constants.

√ (a) Let β and F be fixed. Derive and plot the curve in the (k, l) plane which describes the set of Rossby waves which all have the same frequency, σ.

√ (b) Align Cartesian coordinates so that the x-axis points eastward and the y-axis points northward. Suppose a coastline runs in a straight line, making an angle α with the x-axis, and an ocean occupies the region to the east of the coastline. Consider the problem of oceanic Rossby waves reflecting from the coastline. Write the streamfunction as

$$\psi(x,y,t) = \psi_I(x,y,t) + \psi_R(x,y,t)$$

where

$$\psi_I(x,y,t) \equiv A_I e^{i(k_I x + l_I y - \sigma_I t)}$$
$$\psi_R(x,y,t) \equiv A_R e^{i(k_R x + l_R y - \sigma_R t)}$$

The subscripts I and R refer to the incident and reflected parts of the wave field. Take the boundary condition at the wall to be no flow into or out of the wall. If A_I, k_I, l_I and σ_I are given, find A_R, k_R, l_R, and σ_R. Show all details.

Hint: Use your answer to part (a) as you carefully apply the boundary conditions.

(a)

To derive this relation we make use of the fact that σ is constant.

Then $\frac{\partial\sigma}{\partial k} = \frac{\partial\sigma}{\partial l} = 0$. Let's evaluate $\frac{\partial\sigma}{\partial k}$

$$0 = \frac{\partial\sigma}{\partial k} = \frac{\partial}{\partial k}\left(\frac{-\beta k}{k^2+l^2+F}\right) = \frac{-\beta(k^2+l^2+F) - (-\beta k)(2k)}{(k^2+l^2+F)^2}$$

$$= \frac{-\beta k^2 - \beta l^2 - \beta F + 2\beta k^2}{(k^2+l^2+F)^2}$$

$$0 = \frac{\beta(k^2 - l^2 - F)}{(k^2+l^2+F)^2}$$

This implies

$$0 = \beta(k^2 - l^2 - F)$$

β is a constant not equal to zero. If $\beta = 0$, $\sigma = 0 \rightarrow$ the trivial case.

So we assume β is a nonzero constant. In this case

$$0 = k^2 - l^2 - F$$

or

$$k^2 = l^2 + F$$

or

$$k^2 - l^2 = F \rightsquigarrow \text{hyperbola}$$

Recall the following definition

(i) circle $\rightarrow (x-h)^2 + (y-k)^2 = r^2$ (circle w/ center at (h,k))

(ii) parabola $\rightarrow (x-h)^2 = 4p(y-k)$ vertex at (h,k) with focus p units above the vertex

(iii) ellipse $\rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ center at (h,k)

$a \equiv$ semimajor axis

$b \equiv$ semiminor axis

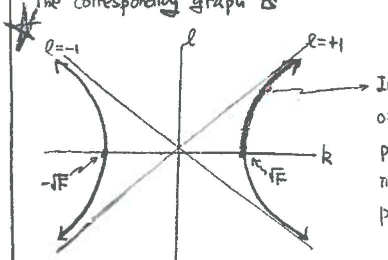
(iv) hyperbola $\rightarrow \frac{\overbrace{(x-h)^2}^{x'}}{a^2} - \frac{\overbrace{(y-k)^2}^{y'}}{b^2} = 1$ with asymptotes $y' = \pm\frac{b}{a}x'$

Thus we see the eq. $k^2 - l^2 = F$ is the eq. for a hyperbola,

$$\boxed{\frac{k^2}{(\sqrt{F})^2} - \frac{l^2}{(\sqrt{F})^2} = 1}$$

with asymptotes $l = \pm\frac{\sqrt{F}}{\sqrt{F}}k = \pm k$

The corresponding graph is



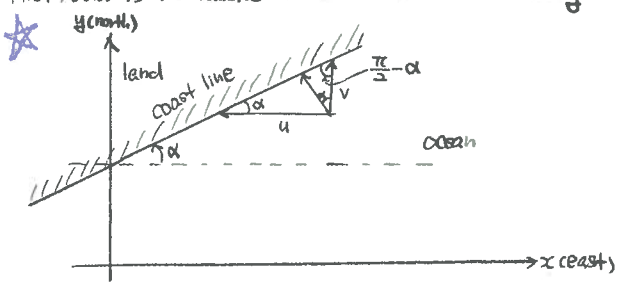
In typical cases the wavenumbers k and l are only defined for positive values, thus the portion of the curve which is of interest is in the 1st quadrant ($k \geq 0$, $l \geq 0$) of the k-l plane.

The curve in the (k, l) plane which describes the set of Rossby waves which all have the same frequency σ is

$$k^2 - l^2 = F.$$

(b)

First, below is a schematic of the situation we are considering



Given the velocity components u and v, what are the boundary conditions on u and v? The BC's are that there is no flow into or out of the coastline. That is, the component of the flow normal to the boundary is zero $\Rightarrow \underline{V} \cdot \hat{n} = 0$

This means that the normal components of u + v at the boundary are zero.

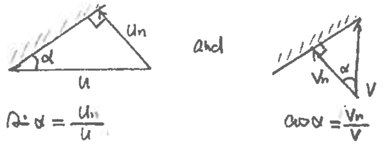
Let u_n and v_n denote the component of u + v along the local normal to the boundary at a point. The BC's tell us

$$\begin{cases} U_n = 0 \\ V_n = 0 \end{cases}$$

Given the stream function ψ, we can define u and v as

$$u = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}$$

How do (U_n, V_n) relate to (u, V) on the boundary? From the diagram we see



$$\sin\alpha = \frac{U_n}{u} \qquad \cos\alpha = \frac{V_n}{V}$$

Thus, $u\sin\alpha = U_n$ and $V\cos\alpha = V_n$

and the boundary conditions on u & V along the coastline become

$$\boxed{u\sin\alpha = 0 \quad \text{and} \quad V\cos\alpha = 0 \qquad BC's}$$

The BC's we are given, namely $\mathbf{V}\cdot\hat{n} = 0$, is a common BC in fluid dynamics problem where we dealing with a fixed solid boundary in an inviscid, nondiffusive fluid. In the current problem, we will assume that there is no damping/amplification of the amplitude of the reflected Rossby waves.

Thus we assume

$$\boxed{A_I = A_R}$$ according to Pedlosky $A_R = -A_I$

The amplitude of the incident and reflected waves is the same. This seems intuitively proper for reflection off a fixed, rigid boundary for waves in an inviscid, nondiffusive fluid.

With regards to the frequency of the reflected Rossby wave it seems intuitive that this frequency should equal that of the incident Rossby wave, particularly under the conditions assumed above. Thus, we take

$$\boxed{\sigma_I = \sigma_R}$$

The frequency of the incident and reflected waves are equal.

This leaves k_R and l_R, the zonal and meridional wavenumbers of the reflected waves. To derive formulae for k_R and l_R we make use of the BC's and the relationship between u, v, ψ

Since $u = -\frac{\partial\psi}{\partial y}$ and $v = \frac{\partial\psi}{\partial x}$ and ψ is given

$$\begin{cases} u = -\frac{\partial\psi}{\partial y} = -i\,l_I\psi_I - i\,l_R\psi_R \\ v = \frac{\partial\psi}{\partial x} = i\,k_I\psi_I + i\,k_R\psi_R \end{cases} \qquad \psi = \psi_I + \psi_R$$

On the boundaries, $\begin{cases} u\sin\alpha = 0 \\ V\cos\alpha = 0 \end{cases}$

Therefore,

$$\begin{cases} -i(l_I\psi_I + l_R\psi_R)\sin\alpha = 0 \\ i(k_I\psi_I + k_R\psi_R)\cos\alpha = 0 \end{cases}$$

We can simplify this by dividing each eq by i. We also note that in general $\sin\alpha \neq 0$, $\cos\alpha \neq 0$. This leaves

$$\begin{cases} l_I\psi_I + l_R\psi_R = 0 \\ k_I\psi_I + k_R\psi_R = 0 \end{cases}$$

We have 2 eqs in 2 unknowns. In principle we can solve for l_R and k_R

Let's substitute for ψ_I and ψ_R

$$\begin{cases} l_I A_I e^{i(k_I x + l_I y - \sigma_I t)} + l_R A_R e^{i(k_R x + l_R y - \sigma_R t)} = 0 \\ k_I A_I e^{i(k_I x + l_I y - \sigma_I t)} + k_R A_R e^{i(k_R x + l_R y - \sigma_R t)} = 0 \end{cases}$$

Make use of the fact that $A_I = A_R$ and $\sigma_I = \sigma_R$

$$\begin{cases} (A_I e^{-i\sigma_I t})(l_I e^{i(k_I x + l_I y)} + l_R e^{i(k_R x + l_R y)}) = 0 \\ (A_I e^{-i\sigma_I t})(k_I e^{i(k_I x + l_I y)} + k_R e^{i(k_R x + l_R y)}) = 0 \end{cases}$$

this is in general $\neq 0$

Let $\phi_I = k_I x + l_I y$, $\phi_R = k_R x + l_R y$

We get

$$\begin{cases} l_I e^{i\phi_I} + l_R e^{i\phi_R} = 0 \\ k_I e^{i\phi_I} + k_R e^{i\phi_R} = 0 \end{cases}$$

or solving for l_R and k_R

$$\boxed{\begin{aligned} l_R &= -l_I\, e^{i(\phi_I - \phi_R)} \\ k_R &= -k_I\, e^{i(\phi_I - \phi_R)} \end{aligned}}$$

In principle this system of 2 trancendental eqs in 2 unknowns can be solved for l_R and k_R.

Recall we have

$$\boxed{\begin{aligned} A_R &= A_I \\ \sigma_R &= \sigma_I \end{aligned}}$$

Thus we have derived formulae for A_R, σ_R, k_R, l_R in terms of the given A_I, σ_I, k_I, l_I

NWP (barotropic vorticity eq.) Holton section 13.4

MET?(Numerical Weather Prediction): Ahlquist (1 hour, 1997) *

- The barotropic vorticity equation is

$\because$ non-divergent $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

$$\frac{\partial \zeta}{\partial t} = -\mathbf{V}\cdot\nabla(\zeta+f) \longrightarrow \frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}(u\zeta)+\frac{\partial}{\partial y}(v\zeta)+\beta v$$

$u=-\frac{\partial \psi}{\partial y}$, $v=\frac{\partial \psi}{\partial x}$

where relative vorticity is $\zeta = \nabla^2\psi$ and the velocity is given by $\mathbf{V} = \mathbf{k}\times\nabla\psi$.

(a) Suppose this equation applies to flow in a channel of width L_y and of length L_x that goes around the Earth. Solid, frictionless walls circle the Earth at $y = 0$ and $y = L_y$.
What is the boundary condition on ψ at $x = 0$ and $x = L_x$?
What is the boundary condition on ψ at $y = 0$ and $y = L_y$?

(b) Notation: Let u'_{jk}, v'_{jk}, ζ'_{jk}, and β_k represent the two velocity components, relative vorticity, and planetary vorticity gradient ($\partial f/\partial y$) at the (j, k)–th grid point at time step t. Write the barotropic vorticity equation in finite difference form using centered differences in time and space with $\Delta x = \Delta y$. Then rearrange this equation so that ζ^{t+1}_{jk} is alone on the left side and all the other terms are on the right. ✓

(c) For your answer to (b), what is the largest value of Δt that will not cause a numerical instability? ✓

(d) Once we have ζ_{jk} at the next time step from (b), we must solve $\nabla^2\psi - \zeta = 0$ to get ψ_{jk}. Write this equation in centered finite difference form assuming $\Delta x = \Delta y$.

(e) Relaxation is one way to solve the finite difference equation in (d). Let $\psi^{(v)}_{jk}$ denote the v–th iteration for ψ_{jk} and let

$$r_{jk} \equiv \nabla^2\psi^{(v)}_{jk} - \zeta_{jk}$$

Denote the residual, i.e., the error, where the right hand side represents the formula you found for (d). With relaxation, we improve an iteration by setting

$$\psi^{(v+1)}_{jk} = \psi^{(v)}_{jk} + c^{(v)}_{jk}$$

where the correction $c^{(v)}_{jk}$ is chosen so that $r_{jk} = 0$ if $\psi^{(v+1)}_{jk}$ is used in the residual formula while ψ at all other points is held at values from the v–th iteration.
Solve for $c^{(v)}_{jk}$.

✓ (f) Weather forecasts based on the barotropic vorticity equation contain long waves that do not travel at the right speed. In a barotropic model, do long waves travel too fast or too slow eastward or westward? How did early forecasters modify the vorticity equation to lessen this problem?

?

$\Rightarrow$ look at Holton ? or other notes $\rightarrow$ O'Brien or Krish's notes!

* The angular momentum budget

ΩR uR ΩR^2

MET?(General circulation): Ahlquist (60 minutes) *

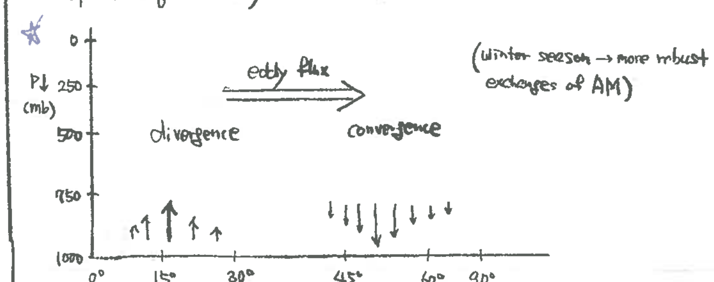
- Compare and contrast the time-averaged maintenance of zonal angular momentum in the atmosphere as viewed from: (i) a zonally-averaged perspective; and (ii) a local perspective, especially in the region of the entrance and exit to the climatological jet stream. A good answer would mention the work of such researchers as Starr, Palmen, Namias and Clapp, Blackmon, and N.-C. Lau. A response from a prospective Ph.D. student could mention directions for current research in this field.

See Holton's section 10.3

Note

In this discussion we will consider the overall balance of angular momentum for the atmosphere and earth combined. Since the average rotation rate of the earth itself is observed to be very close to constant, the atmosphere must also on average conserve its angular momentum. The atmosphere gains angular momentum (AM) from the earth in the tropics where the surface winds are easterly and gives up AM to the earth in the mid latitudes where the surface winds are westerly. Thus, there must be a net poleward transport of AM within the atmosphere. The AM given by the earth to the atmosphere in the belt of easterlies must just balance the AM given up to the earth in the westerlies if the global angular momentum of the atmosphere is to remain constant.

In the equatorial regions the poleward momentum transport is divided between advection of absolute AM by the poleward flow in the axially symmetric Hadley circulation and transport by eddies. In the mid-lats, however, it is primarily the eddy motions that transport momentum poleward. The AM budget of the atmosphere is qualitatively shown below.



In the winter season there is a maximum poleward flux of AM at around 30° latitude and a max horizontal flux convergence at about 45°. This maximum in the flux convergence is a reflection of the strong energy conversion in the upper-level westerlies (ZAPE → EAPE → EKE) and is the mechanism whereby the atmosphere can maintain a positive zonal wind in the mid-lats, despite the momentum loss to the sfc.

It is convenient to analyze the momentum budget in terms of absolute AM since this is conserved for an air parcel in the absence of frictional or pressure torques. The absolute AM per unit mass of the atmosphere is

$$\boxed{M = (\Omega a\cos\phi + u)\,a\cos\phi} \qquad (1)$$

tangential speed of earth at lat ϕ — tangential speed of parcel — arm length at latitude ϕ

or

$$M = R^2\Omega + Ru$$

where $R = a\cos\phi \equiv$ arm length at lat. ϕ where a is the radius of the earth.

The absolute AM of an individual air parcel can be changed only by torques caused by the zonal pressure gradient and the eddy stress. In isobaric coordinates we have

$$\boxed{\frac{dM}{dt} = -a\cos\phi\left[\frac{\partial\Phi}{\partial x} + g\frac{\partial\tau_E}{\partial p}\right]} \qquad (2)$$

or

$$\frac{dM}{dt} = -R\left(\frac{\partial\Phi}{\partial x}\right) - R(F_\lambda)$$

τ_E^x is the zonal component of the vertical eddy stress. We assume the horizontal eddy stresses are negligible compared to the vertical eddy stress.

We find it more convenient to discuss the AM budget when the eqs are cast into the sigma coord. system. In the σ coord. system, the vertical coord. is a nondimensional number whose value decreases from 1 at the lower boundary to 0 at the top of the atm. A common definition of σ that satisfies these conditions is

$$\sigma = \frac{P}{P_s}$$

where P_s is the sfc pressure. The vertical velocity in σ coordinates is defined by

$$\dot\sigma = \frac{d\sigma}{dt} \quad ; \quad \dot\sigma = 0 \text{ at } \sigma = 1$$

and will always be zero at the lower boundary regardless of the terrain

↳ $\sigma = 1$ at the lower boundary and $\frac{d}{dt}(\text{constant}) = 0$

We may transform the AM eq (2) from isobaric to σ coordinates with the result being

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\nabla + \dot\sigma\frac{\partial}{\partial\sigma}\right)M = -a\cos\phi\left(\left(\frac{\partial\Phi}{\partial x} + \frac{RT}{P_s}\frac{\partial P_s}{\partial x}\right) + \frac{g}{P_s}\frac{\partial\tau_E^x}{\partial\sigma}\right) \qquad (3)$$

∵ In σ-coord. the isobaric pressure gradient

$$\nabla_p\Phi = \nabla_\sigma\Phi - \sigma\nabla_\sigma\ln P_s\frac{\partial\Phi}{\partial\sigma}$$

hydrostatic eq in σ

$$= \nabla_\sigma\Phi - \sigma\nabla_\sigma\ln P_s\left(-\frac{RT}{\sigma}\right)$$

$$= \nabla_\sigma\Phi - \sigma\left(-\frac{1}{\sigma}\right)RT\nabla_\sigma\ln P_s$$

$$= \nabla_\sigma\Phi + \frac{RT}{P_s}\left(\frac{\partial P_s}{\partial x}\mathbf{i} + \frac{\partial P_s}{\partial y}\mathbf{j}\right)$$

$$= \left(\frac{\partial\Phi}{\partial x} + \frac{RT}{P_s}\frac{\partial P_s}{\partial x}\right)\mathbf{i} + \left(\qquad\right)\mathbf{j}$$

$$\therefore \left.\frac{\partial\Phi}{\partial x}\right|_p = \left.\left(\frac{\partial\Phi}{\partial x} + \frac{RT}{P_s}\frac{\partial P_s}{\partial x}\right)\right|_\sigma$$

With the aid of the continuity eq. in σ coord,

$$\frac{\partial P_s}{\partial t} + \nabla\cdot P_s\underset{\sim}{V} + P_s\frac{\partial\dot\sigma}{\partial\sigma} = 0 \qquad (4)$$

We can rewrite (3) in flux form as

$$\frac{\partial(P_sM)}{\partial t} = -\nabla\cdot(P_sM\underset{\sim}{V}) - \frac{\partial}{\partial\sigma}(P_sM\dot\sigma) - a\cos\phi\left[P_s\frac{\partial\Phi}{\partial x} + RT\frac{\partial P_s}{\partial x}\right] - g\,a\cos\phi\frac{\partial\tau_E^x}{\partial\sigma} \qquad (5)$$

Now we write

$$\nabla\cdot(P_sM\underset{\sim}{V}) = \frac{\partial}{\partial x}(P_sMu) + \frac{1}{\cos\phi}\frac{\partial(P_sMv\cos\phi)}{\partial y}$$

$$P_s \frac{\partial\Phi}{\partial x} + RT\frac{\partial P_s}{\partial x} = P_s\frac{\partial}{\partial x}(\Phi - RT) + \frac{\partial}{\partial x}(P_s RT) = P_s\frac{\partial}{\partial x}\left(\frac{\partial}{\partial\sigma}(\sigma\Phi)\right) + \frac{\partial}{\partial x}(P_s RT)$$

∵ via hydrostatic eq. $\frac{\partial\Phi}{\partial\sigma} = -\frac{RT}{\sigma}$ so $-RT = \sigma\frac{\partial\Phi}{\partial\sigma}$

$$\Phi - RT = \Phi + \sigma\frac{\partial\Phi}{\partial\sigma} = \frac{\partial}{\partial\sigma}(\sigma\Phi)$$

$$= \frac{\partial}{\partial\sigma}\left(P_s\sigma\frac{\partial\Phi}{\partial x}\right) + \frac{\partial}{\partial x}(P_sRT)$$ since P_s does not depend on σ

We therefore rewrite (5) as

$$\frac{\partial(P_sM)}{\partial t} = -\frac{\partial}{\partial x}(P_sMu) - \frac{1}{\cos\phi}\frac{\partial}{\partial y}(P_sMv\cos\phi) - a\cos\phi\left(\frac{\partial}{\partial\sigma}\left(P_s\sigma\frac{\partial\Phi}{\partial x}\right) + \frac{\partial}{\partial x}(P_sRT)\right) - \frac{\partial}{\partial\sigma}(P_sM\dot\sigma) - ga\cos\phi\frac{\partial\tau_E^x}{\partial\sigma}$$

$$= -\frac{\partial}{\partial x}(P_sMu) - \frac{1}{\cos\phi}\frac{\partial}{\partial y}(P_sMv\cos\phi) - \frac{\partial}{\partial\sigma}\left(P_sM\dot\sigma + ga\cos\phi\,\tau_E^x + a\cos\phi\,P_s\sigma\frac{\partial\Phi}{\partial x}\right) - a\cos\phi\frac{\partial}{\partial x}(P_sRT)$$

Zonally average this eq. All $\frac{\partial}{\partial x}$ terms zonally average to zero since they are perfect differentials. We are left with

$$\frac{\partial}{\partial t}[P_sM] = -\frac{1}{\cos\phi}\frac{\partial}{\partial y}[P_sMv\cos\phi] - \frac{\partial}{\partial\sigma}\left([P_sM\dot\sigma] + ga\cos\phi[\tau_E^x] + a\cos\phi\left[\sigma P_s\frac{\partial\Phi}{\partial x}\right]\right) \qquad (6)$$

← brackets now mean zonal average

Now integrate (6) from the sfc ($\sigma=1$) to the top of the atmosphere ($\sigma=0$)

$$\int_0^1 \frac{\partial}{\partial t}[P_sM]d\sigma = -(\cos\phi)^{-1}\int_0^1\frac{\partial}{\partial y}([P_sMv]\cos\phi)d\sigma - \overset{\to 0}{[P_sM\dot\sigma]\Big|_{\sigma=0}^{1}} - ga\cos\phi[\tau_E^x]\Big|_0^1 - a\cos\phi\left[\sigma P_s\frac{\partial\Phi}{\partial x}\right]\Big|_0^1 \qquad (7)$$

Recall the BC $\dot\sigma = 0$ at $\sigma=0$ & $\sigma=1$

$$\therefore [P_sM\dot\sigma]\Big|_0^1 = 0$$

Next assume the eddy stress at $\sigma=0$ is negligible so

$$+ga\cos\phi[\tau_E^x]\Big|_0^1 = ga\cos\phi[\tau_E^x]_{\sigma=1}$$

Lastly, $a\cos\phi\left[\sigma P_s\frac{\partial\Phi}{\partial x}\right]\Big|_0^1 = a\cos\phi\left[P_s\frac{\partial\Phi}{\partial x}\right]_{\sigma=1}$

Define $h(x,y) = \frac{1}{g}\Phi(x,y,1)$

Then

$$\left[P_s\frac{\partial\Phi}{\partial x}\right]\Big|_{\sigma=1} = \left[P_s\cdot g\cdot\frac{1}{g}\frac{\partial\Phi}{\partial x}\right]\Big|_{\sigma=1} = \left[P_s g\frac{\partial}{\partial x}\left(\frac{\Phi}{g}\right)\right]\Big|_{\sigma=1} = g\left[P_s\frac{\partial h}{\partial x}\right]\Big|_{\sigma=1}$$

With the above points noted we divide (7) by g and obtain

$$\boxed{\int_0^1 g^{-1}\frac{\partial}{\partial t}[P_sM]d\sigma = -(ga\cos\phi)^{-1}\int_0^1\overset{①}{\frac{\partial}{\partial y}([P_sMv]\cos\phi)}d\sigma - a\cos\phi\left(\underset{②}{[\tau_E^x]\Big|_{\sigma=1}} + \underset{③}{\left[P_s\frac{\partial h}{\partial x}\right]\Big|_{\sigma=1}}\right)} \qquad (8)$$

This eq expresses the AM budget for a zonal ring of air of unit meridional width, extending from the ground to the top of the atmosphere.

The 3 terms on the RHS represent

- ① ≡ convergence of the meridional AM flux
- ② ≡ torque due to small scale turbulent fluxes at the sfc
- ③ ≡ the surface pressure torque

As the AM of the earth-atmosphere system is roughly constant, in the long-term mean

$$\int_0^1 g^{-1}\frac{\partial}{\partial t}[P_sM]d\sigma = 0$$

Thus the 3 terms on the RHS must balance.

Sfc pressure torque

The pressure torque term, $\left[P_s\frac{\partial h}{\partial x}\right]_{\sigma=1}$, acts to transfer AM from the atmosphere to the ground provided that the sfc pressure and slope of the ground are positively correlated.

$$\int_0^1\frac{1}{g}\frac{\partial}{\partial t}[P_sM]d\sigma \propto -a\cos\phi\left[P_s\frac{\partial h}{\partial x}\right]_{\sigma=1}$$ → when this is positive the RHS<0 which implies atmospheric AM↓

Observations indicate that P_s and $\frac{\partial h}{\partial x}$ are generally positively correlated in the mid lats. In mid-lats there is a slight tendency for the sfc pressure to be higher on the western side of mountains than on the eastern sides. The eastern sides typi[cally] have lower pressure (lee troughs) which correlates positively with the decreasing sf[c] slope. Thus $\left[P_s\frac{\partial h}{\partial x}\right]_{\sigma=1} > 0$ in mid-lats.

Sfc eddy stress

In the mid-lats of the N.H the sfc pressure torque produces nearly 1/2 of the total atmosphere-sfc exchange. In the tropics and the S.H the exchange (at th[e] sfc) is dominated by turbulent eddy stresses (term ②)

Eddy flux

The role of eddy motions in providing the meridional AM transport necessary to bala[nce] the sfc AM sinks can be best demonstrated if we partition the flow into a zonal mean and deviations from the mean. We let

$$M = [M] + M' = (\Omega a\cos\phi + [u] + u')a\cos\phi$$

$$P_sv = [P_sv] + (P_sv)'$$

∵ [ ] ≡ zonal mean, ′ ≡ deviation from zonal mean

Thus, the meridional flux becomes

$$[P_sMv] = [([M]+M')([P_sv] + (P_sv)')]$$

$$= [((\Omega a\cos\phi + [u] + u')a\cos\phi)([P_sv] + (P_sv)')]$$

$$= [\Omega a^2\cos^2\phi[P_sv]] + [\cancel{\Omega a^2\cos^2\phi(P_sv)'}]$$ → the mean of a perturbation is zero.

$$+ [\cancel{[u]a\cos\phi(P_sv)'}] + [[u]a\cos\phi[P_sv]]$$

$$+ [\cancel{u'a\cos\phi[P_sv]}] + [u'a\cos\phi(P_sv)']$$

$$= \left(\underset{①}{\Omega a\cos\phi[P_sv]} + \underset{②}{[u][P_sv]} + \underset{③}{[u'(P_sv)']}\right)a\cos\phi$$

The 3 terms on the RHS are the Ω-momentum flux, the drift, and the eddy momentum flux, respectively

Term ①

We can show that the Ω-momentum flux does not contribute to the vertically integrated flux.

Term ②

The drift term is important in the tropics but in the mid-lats it is small compare[d] to the eddy flux. In the long time mean the drift term is negligible (effective[ly] zero). To see this we zonally average the continuity eq.

$$\frac{\partial P_s}{\partial t} = -\frac{\partial}{\partial x}(P_su) - \frac{1}{\cos\phi}\frac{\partial}{\partial y}(P_sv\cos\phi) - P_s\frac{\partial\dot\sigma}{\partial\sigma}$$

to get

$$\frac{\partial[P_s]}{\partial t} = 0 - \frac{1}{\cos\phi}\frac{\partial}{\partial y}[P_sv\cos\phi] - \left[P_s\frac{\partial\dot\sigma}{\partial\sigma}\right]$$

Next, vertically integrate from the sfc to the top of the atmosphere

$$\int_0^1\frac{\partial[P_s]}{\partial t}d\sigma = -\frac{1}{a\cos\phi}\frac{\partial}{\partial y}\int_0^1[P_sv]\cos\phi\, d\sigma - \cancel{\int_0^1 P_s\frac{\partial\dot\sigma}{\partial\sigma}d\sigma}$$ → 0 since $\dot\sigma=0$ at $\sigma=1$ & $\sigma=0$

Since P_s is independent of σ and we get $\frac{\partial[P_s]}{\partial t}(1-0)$

or $$\frac{\partial[P_s]}{\partial t} = -(\cos\phi)^{-1}\frac{\partial}{\partial y}\int_0^1[P_sv]\cos\phi\, d\sigma$$

For a long term time averaged flow the LHS of the above eq vanishes (we do not see P_s ↑ or ↓ systematically over a long time period). This implies that ove[r] a long time average there is no net meridional mass flux across latitude circle[s]. Thus, in a long time average sense the drift term vanishes.

Term ③

The eddy flux of momentum is the dominant term in a long time mean.

From above discussion we conclude that the meridional flux may be approx. a[s]

$$[P_sMv] \simeq [u'(P_sv)']a\cos\phi$$

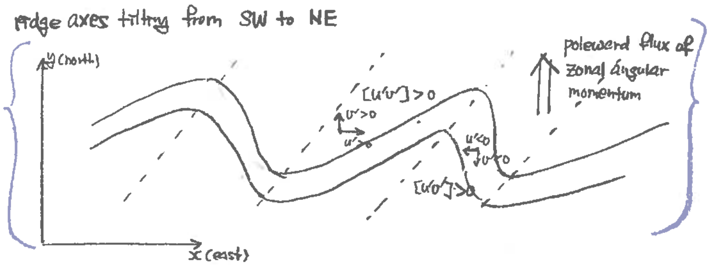
∵ assume the fractional change in P_s is small compared to the change in v' so $(P_sv)' \sim [P_s]v'$

Therefore

$$(-ga\cos\phi)^{-1}\int_0^1\frac{\partial}{\partial y}([P_sMv]\cos\phi)d\sigma \simeq -(ga\cos\phi)^{-1}\int_0^1\frac{\partial}{\partial y}(a\cos^2\phi[P_s][u'v'])d\sigma$$

and we arrive at the conclusion that the eddy flux of momentum must contribute significantly to the AM budget.

In the N.H., the eddy momentum flux is positive and decreasing with latitude in the belt of the westerlies. For QG flow, positive eddy momentum flux requires that the eddies be asymmetric in the horizontal plane with the trough and ridge axes tilting from SW to NE



When the troughs and ridges on average have SW to NE phase tilt, the zonal flow will be larger than average ($u' > 0$) where the meridional flow is poleward ($v' > 0$) and less than average ($u' < 0$) where the meridional flow is equatorward ($v' < 0$). Thus $[u'v'] > 0$ and the eddies systematically transport positive zonal momentum poleward.

As shown in (6) the total momentum flux consists of flux due to

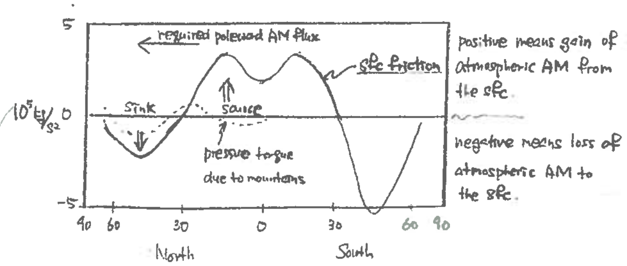
① large scale motions. $[P_s M \dot{\sigma}]$?

② the pressure torque, $a\cos\phi[p\frac{\partial\Phi}{\partial x}]$

③ Small scale turbulent stresses, $ga\cos\phi[T_E^x]$

As previously mentioned, the last two are responsible for the transfer of momentum from the earth to the atmosphere in the tropics and the atmosphere to the earth in the mid-lats. Outside the PBL, however, the vertical momentum transport in the troposphere is primarily due to the Ω-momentum flux $\Omega a\cos\phi[P_s\dot{\sigma}]$.

Estimates of the torques due to turbulent transfer from the sfc and the pressure torque due to large scale topography have been attempted by several investigators. One such estimate of the latitudinal variation of the eastward torque exerted by the earth on the atmosphere is shown below.

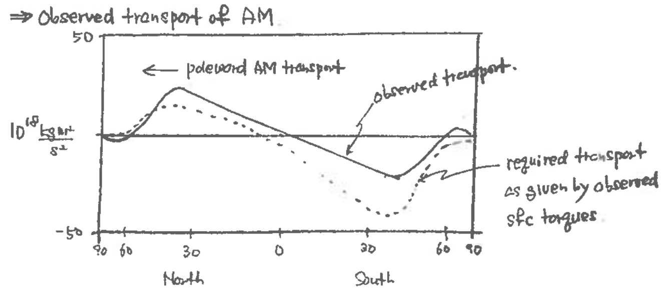


< Based on Lorenz, 1967 >

average eastward torque per unit horizontal area exerted on the atmosphere by sfc friction + mountains

The northward flux of AM required to balance the total sfc torque is shown below. This flux can be directly estimated from wind data

⇒ Observed transport of AM



It should be mentioned that except for the belt within 10° of the equator, almost all the northward flux is due to the eddy flux term $\overline{u'v'}$.

MET? : Ahlquist (15 minutes)

- Question

(a) Describe the "Perfect Prog" method and the method of "Model Output Statistics" in sufficient detail to show the difference between the two.

(b) What are the relative advantages and disadvantages (if any) for the two methods?

(c) Describe how one of these two methods has been used to predict probability of precipitation (POP).

?

I don't know!

V

* Fluid flow from a faucet

MET? (Fundamental dynamics): Ahlquist (60 minutes) **

• When water flows from a faucet, the radius, r, of the stream of water decreases as the distance, z, from the faucet increases.

a) List as many physical mechanisms as you can that might affect the radius of the stream of water from a faucet.

b) In your opinion, what is the most important reason why r decreases as z increases? Use that physical principle to derive a formula for r as a function of z and any other relevant parameters.

c) Describe how to do an experiment using your kitchen sink and common household items which would test the theory you derived in the previous step.

?

a)
- diameter of the faucet where the water originated
- rate of flow of the water out of the faucet
- surface tension of water