1 Theory of the Hadley circulation

1.1 Angular momentum conservation: implications

We will begin our attempts to understand the "big picture" of the structure of the atmosphere by asking about what theory predicts if we ignore eddies and assume that the atmosphere and its circulation are two-dimensional. Of course, in reality the atmosphere is full of eddies but, as we'll see, we can get a handle on some important, and relevant, concepts by ignoring them in the first instance. Later in the class, we will come to assess the impact of the eddies.

The 2D governing equation of zonal motion can be written

$$\rho\frac{dm}{dt} = \rho\left(\frac{\partial m}{\partial t} + \mathbf{u}\cdot\nabla m\right) = \rho\left(\frac{\partial m}{\partial t} + \frac{v}{a}\frac{\partial m}{\partial \varphi} + w\frac{\partial m}{\partial z}\right) = G \ , \qquad (1)$$

where a is the Earth radius, φ is latitude, $m = \Omega a^2\cos^2\varphi + ua\cos\varphi$ is the absolute angular momentum[1] (about the rotation axis) per unit mass, and G represents frictional and other torques. If $G = 0$, m is conserved, a simple statement of angular momentum conservation for axisymmetric flow. If G represents friction, we may write[2]

$$G = \nabla\cdot(\nu\rho\nabla m) \ .$$

The continuity equation is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \ ,$$

whence we can rewrite (1) as

$$\frac{\partial}{\partial t}(\rho m) = -\nabla\cdot(\mathbf{F}_{adv} + \mathbf{F}_{fric})$$

where the advective flux is

$$\mathbf{F}_{adv} = \rho\mathbf{u}m$$

[1] The absolute zonal velocity about the rotation axis is $u + \Omega r$, where Ω is the earth's rotation rate, u the zonal velocity and $r = a\cos\varphi$. So the angular momentum per unit mass is

$$m = r(u + \Omega r) = \Omega a^2\cos^2\varphi + ua\cos\varphi \ .$$

[2] This is not generally true (*i.e.,* viscous diffusion in latitude does not homogenize angular momentum) but since vertical scales $\ll$ horizontal scales, only the vertical components matter, in which case

$$G = a\cos\varphi\frac{\partial}{\partial z}\left(\nu\rho\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}\left(\nu\rho\frac{\partial m}{\partial z}\right) \simeq \nabla\cdot(\nu\rho\nabla m) \ .$$

and the frictional flux is

$$\mathbf{F}_{fric} = -\rho\nu\nabla m \ .$$

Now, suppose in steady state there is an extremum of m somewhere other than at the surface, as illustrated (for the case of a maximum) in Fig. 1. Consider first the mass budget within the shaded area $\mathcal{A}$

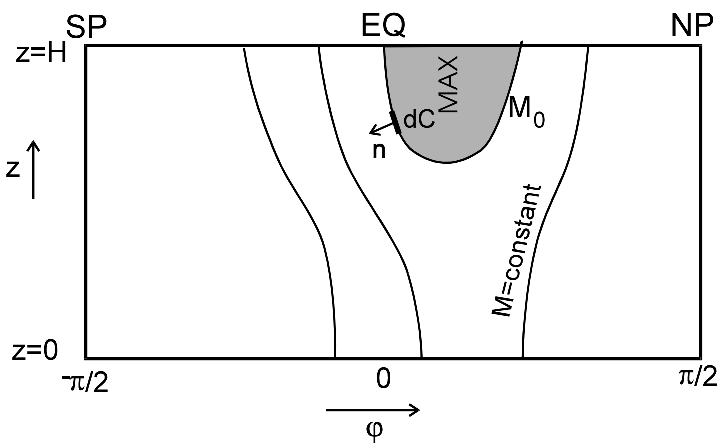


Figure 1: Extremum of m.

enclosed by the contour $\mathcal{C}$ on which $m = M_0$; integrating the continuity equation

$$\iint\limits_{\mathcal{A}} \frac{\partial\rho}{\partial t}\ dA + \iint\limits_{\mathcal{A}} \nabla\cdot\rho\mathbf{u}\ dA = \iint\limits_{\mathcal{A}} \frac{\partial\rho}{\partial t}\ dA + \oint_{\mathcal{C}} \rho\mathbf{u}\cdot\mathbf{n}\ dl\ = 0\ ,$$

where $\mathbf{n}$ is the unti normal out of $\mathcal{C}$. In steady state, therefore,

$$\oint_{\mathcal{C}} \rho\mathbf{u}\cdot\mathbf{n}\ dl\ = 0\ , \tag{2}$$

which simply states that there is no net mass flux into or out of the shaded region if the mass within the region is constant. Now, consider the angular momentum budget; the net *advective* flux of m across $\mathcal{C}$ is

$$\oint_{\mathcal{C}} \mathbf{F}_{adv}\cdot\mathbf{n}\ dl = \oint_{\mathcal{C}} \rho\mathbf{u}\cdot\mathbf{n}m\ dl\ .$$

But, since m is constant on $\mathcal{C}$,

$$\oint_{\mathcal{C}} \rho\mathbf{u}\cdot\mathbf{n}m\ dl = M_0\oint_{\mathcal{C}} \rho\mathbf{u}\cdot\mathbf{n}\ dl = 0\ ,$$

by (2): the net advective flux of m across a contour of m is *exactly* zero. However, the net frictional flux out of the shaded region is

$$\oint_C \mathbf{F}_{fric} \cdot \mathbf{n}\, dl = -\oint_C \rho\nu\mathbf{n} \cdot \nabla m\, dl > 0 \ ,$$

since along the contour $\mathbf{n} \cdot \nabla m < 0$, by definition (the contour encloses a maximum), and the downgradient viscous fluxes must tend to reduce the maximum im m. So there is a viscous loss of m which —*no matter how small viscosity is* — cannot be balanced by advection if there is an extremum of m anywhere except on a lower boundary (where there can be viscous forces on the boundary to give balance). This leads to **Hide's theorem**: *in steady state, there can be no extrema of angular momentum except at the lower boundary.*

The second important conclusion we can draw, more directly, from (1) is that in steady flow, if friction is negligible, $G = 0$ and hence

$$\mathbf{u} \cdot \nabla m = 0 \ . \qquad (3)$$

This simply states that, since there is no frictional flux, the advective flux must be nondivergent. Put another way, *in an inviscid steady state, there can be no flow crossing angular momentum contours.* Thus has simple and important implications.

1.2 The Held-Hou theory for equatorial symmetry

[The original paper on this is: I. M. Held & A.Y. Hou, *J. Atmos. Sci.*, **37**, 515-533, 1980.]

Consider the steady response of an axisymmetric, *Boussinesq*, spherical atmosphere, inviscid except near its lower boundary, to axisymmetric thermal forcing. The domain is illustrated in Fig. 2; we assume a rigid no-slip lower boundary on $z = 0$: a rigid stress-free boundary on $z = H$.

The eastward and northward momentum eqs., with vertical viscosity, are

$$\begin{aligned} \frac{\partial u}{\partial t} + (\mathbf{v} \cdot \nabla)u - \frac{\tan\ \varphi}{a}uv - 2\Omega\ sin\varphi\ v &= \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right) \\ \frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)v + \frac{\tan\ \varphi}{a}u^2 + 2\Omega\ sin\varphi\ u &= -\frac{1}{a}\frac{\partial \Phi}{\partial \varphi} + \frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right) \end{aligned} \qquad (4)$$

(The first of these — the zonal equation of motion — can be derived directly from the angular momentum budget (1). The second is just

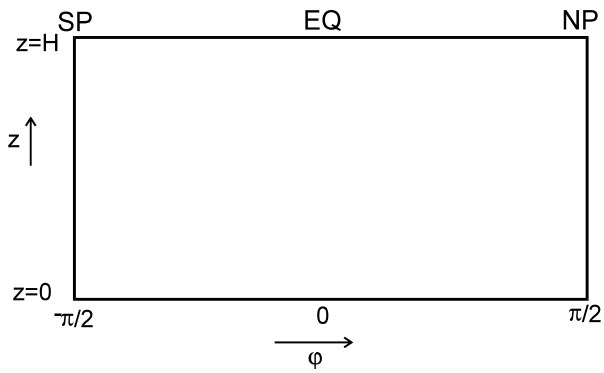


Figure 2: The domain.

the usual northward component of the equation of motion on a rotating sphere, with the Coriolis term and the added centrifugal term: the centrifugal acceleration in a direction normal to the rotation axis is $u^2/r = u^2/\left(a\cos\varphi\right)$, and its northward component is $-\sin\varphi$ times this.)

The continuity equation for 2D flow on the sphere is[3]

$$\nabla\cdot\mathbf{u} = \frac{1}{acos\varphi}\frac{\partial}{\partial\varphi}(v\ cos\varphi) + \frac{\partial w}{\partial z} = 0\ . \tag{5}$$

Hydrostatic balance: in the Boussinesq approximation, $\rho = p/RT \simeq (p/RT_0)\left(1 - T'/T_0\right)$, where $T = T_0 + T'$ and T_0 is a uniform reference temperature. So $\rho_0 = p/RT_0$ and $\rho'/\rho_0 = -T'/T_0$, and buoyancy is $b = gT'/T_0$. Hence the hydrostatic equation is

$$\frac{\partial\Phi}{\partial z} = g\frac{T'}{T_0}\ . \tag{6}$$

This system is subjected to a simple Newtonian representation of diabatic effects, to mimic the latitudinal gradient of solar flux; T is simply relaxed toward a "radiative equilibrium" temperature[4] $T_e\left(\varphi, z\right)$ at a rate α:

$$\frac{\partial T'}{\partial t} + \frac{v}{a}\frac{\partial T'}{\partial\varphi} + wS = Q = -\alpha(T - T_e)\ , \tag{7}$$

[3] The divergence operator in spherical geometry (λ, φ, z) is

$$\nabla\cdot\mathbf{u} = \frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(v\cos\varphi\right) + \frac{\partial w}{\partial z}\ .$$

[4] So $T_e(\varphi, z)$ is the temperature the atmosphere would have in a steady state with no fluid motions to transport heat.

where the static stability is $S = \partial T/\partial z + g/c_p$.

We will assume **nonlinear balance**[5]. In this context, this amounts to assuming that the zonal velocity is much greater than the meridional components (this can be justified *a posteriori*), so $|u| >> |v|$, and $|u| >> |w|L/H$. Also, we assume $|\partial v/\partial t| << \max[f|u|, |u|^2/a]$ (in any case, we are going to consider steady solutions), and neglect the viscous term on v. Then the second of (4) gives the nonlinear balance condition

$$2\Omega\ sin\varphi\ u + \frac{\tan\ \varphi}{a} u^2 = -\frac{1}{a}\frac{\partial \Phi}{\partial \varphi}\ . \tag{8}$$

This just geostrophic balance in the northward direction, with the addition of the centrifugal term (and, since we cannot assume small Rossby number in the tropics, we have no basis to neglect the latter.) Using hydrostatic balance gives the nonlinear thermal wind relation

$$\frac{\partial}{\partial z}\left(2\Omega\ sin\varphi\ u + \frac{\tan\ \varphi}{a} u^2\right) = -\frac{g}{aT_0}\frac{\partial T}{\partial \varphi}\ . \tag{9}$$

Now, neglect viscosity ν in (4) — except at the lower boundary — and consider the steady balances. We have seen from (3) that $\mathbf{u}\cdot\nabla m = 0$ in such circumstances; therefore, on the model top, $z = H$, where $w = 0$, **EITHER**:

(i) $v = 0$; $\rightarrow$ $w = 0$; $\rightarrow$ $Q = 0$; $\rightarrow$ $T = T_e$— this is the thermal equilibrium (TE) regime $\rightarrow$ no meridional circulation$\rightarrow$ no diabatic heating, **OR**

(ii) $\partial m/\partial\varphi = 0$ on the top boundary — this is the angular momentum conserving (AMC) regime, in which there can be a nonzero meridional circulation and hence $T \neq T_e$.

Which is the the correct inviscid limit? The AMC regime cannot go all the way to the poles, since constant $m = \Omega a^2 cos^2\varphi + ua\ cos\varphi$ implies $u \rightarrow \infty$ at the poles. But can the TE regime exist everywhere? Suppose it does—look at the TE solution. $T = T_e$, so $u = u_e(\varphi, z)$ where

$$\frac{\partial}{\partial z}\left(2\Omega\ sin\varphi\ u_e + \frac{\tan\ \varphi}{a} {u_e}^2\right) = -\frac{g}{aT_0}\frac{\partial T_e}{\partial \varphi}\ .$$

Take a separable T_e distribution of the form $T_e = T_{00}(z) + T_o\ \Theta(\varphi)Z(z)$; assume $u = 0$ on $z = 0$; then

$$2\Omega\ sin\varphi\ u_e + \frac{\tan\ \varphi}{a} {u_e}^2 = -\frac{g}{a}\frac{d\Theta}{d\varphi}\int_0^z Z(z')dz'\ .$$

[5] This is just equivalent to the usual balance assumption that the nondivergent flow >>irrotational flow. The nondivergent flow is $(u, 0)$ (since $\partial u/\partial\lambda = 0$); the irrotational flow is $(0, v)$ (since $\partial v/\partial\lambda = 0$).

So the TE solution is untenable if:

(i) if $\partial\Theta/\partial\varphi$ is nonzero at equator: this would give $u \to \infty$ there.

(ii) invoking Hide's theorem, the TE solution is unreachable if the absolute angular momentum $m = \Omega a^2 \cos^2 \varphi + u_e a \cos\varphi$ has any extrema away from the bottom. It turns out that this constraint is violated if T_e has any latitudinal curvature at the equator. Specifically, suppose T_e is a maximum on the equator (setting $Z > 0$, for definiteness) then Θ has a maximum on the equator, so $\Theta_\varphi(0) = 0$. Writing $\Theta_{\varphi\varphi}(\varphi = 0) = -\gamma$ (where $\gamma > 0$) then, near the equator., $d\Theta/d\varphi \simeq -\gamma\ \varphi$. Hence, near $\varphi = 0$,

$$2\Omega\ u_e + \frac{1}{a}{u_e}^2 = \frac{g}{a}\gamma \int_0^z Z(z')dz'$$

so

$$u_e = -\Omega a \pm \left(\Omega^2 a^2 + g\gamma \int_0^z Z(z')dz'\right)^{1/2} .$$

The physically reasonable solution has the + sign[6]. This solution has westerlies on the equator, increasing with z. This implies a maximum of m on the equator, which is untenable: if T_e has this kind of distribution (with negative curvature at the equator) then the inviscid TE solution constitutes a singular limit.

So, in summary, we must have the AMC solution in low latitudes (if T_e maximum there with nonzero $\partial^2 T_e/\partial\varphi^2$), and the TE solution in high latitudes. We need to match the solutions to get the full inviscid solution.

1.2.1 Solution for the cell boundaries [G]

In both TE and AMC regions, integrate (9) in the vertical, and assume $u(\varphi, 0) << u(\varphi, H)$ (because of surface drag), whence

$$2\Omega\ \sin\varphi\ u(\varphi, H) + \frac{\tan\ \varphi}{a} u^2(\varphi, H) = -\frac{gH}{aT_0}\frac{d}{d\varphi}\langle T\rangle\ ,$$

where

$$\langle T\rangle = \frac{1}{H}\int_0^H T\ dz$$

[6]The minus sign gives, in the limit $\gamma \to 0$, $u = -2\Omega a$: relative to inertial space, the atmosphere rotates at the same speed as the solid planet, but in the opposite direction! It seems safe to reject this solution as unphysical.

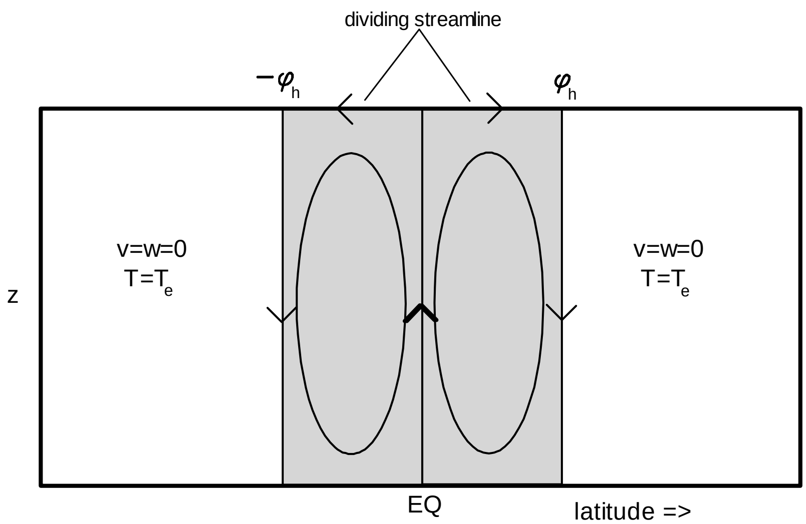


Figure 3: Schematic. AMC region is shaded, TE unshaded. (Note that the edges of the Hadley cell do not need to be vertical.)

is the vertically averaged temperature. In the AMC region, $m = constant = M_0$, say, along the top streamline, whence

$$u(\varphi, H) = u_m = \frac{1}{a\ \cos\varphi}(M_0 - \Omega a^2 \cos^2\varphi)\ .$$

For cases *symmetric* about the equator, expect the meridional circulation to look as shown in Fig. 3, with two cells, one on either side of the equator. In the rising branch, the dividing streamline flow comes up out of boundary layer at the equator where we assume $u = 0$; if m is conserved in the updraft, then must have $M_0 = \Omega a^2$. Then

$$u_m = \Omega a \frac{\sin^2\varphi}{\cos\ \varphi}\ .$$

In the AMC region, $u(\varphi, H) = u_m$ and $T = T_m$ where

$$\begin{aligned}\frac{gH}{aT_0}\frac{d}{d\varphi}\langle T_m\rangle &= -2\Omega\ \sin\varphi\ u_m - \frac{\tan\ \varphi}{a}{u_m}^2 \\ &= -\frac{1}{2}\Omega^2 a \frac{d}{d\varphi}\left(\frac{\sin^4\varphi}{\cos^2\varphi}\right)\ .\end{aligned}$$

So

$$\langle T_m\rangle(\varphi) - \langle T_m\rangle(0) = -\frac{\Omega^2 a^2}{2gH}T_0\frac{\sin^4\varphi}{\cos^2\varphi} \qquad (10)$$

in the AMC region. Thus, the structure of $\langle T_m \rangle$ across the AMC cell is dictated entirelyby the dynamics of amgular momentum conservation, and is independet of the structure of the structure of T_e.

In the TE regime, $T = T_e$. Since Φ must be continuous in latitude and

$$\langle T \rangle = \frac{1}{H} \int_0^H T \, dz = \frac{T_0}{gH} \int_0^H \frac{\partial \Phi}{\partial z} \, dz = \frac{T_0}{gH} \left[\Phi(\varphi, H) - \Phi(\varphi, 0) \right] \ ,$$

$\langle T \rangle$ must be continuous across the edge of the cells at $\varphi = \varphi_h$ where

$$< T_m > (\varphi_h) =< T_e > (\varphi_h) \ .$$

We need a second matching condition [there are 2 unknowns: φ_h and $< T_m > (0)$]. We get this from the steady thermodynamics equation; integrating (7) over the globe:

$$\int_0^H \int_0^{\frac{\pi}{2}} Q \ cos\varphi \ d\varphi \ dz = 0$$

whence, since $Q = -\alpha (T - T_e) = 0$ in $|\varphi| > |\varphi_h|$,

$$\int_0^{\varphi_h} < T_m > \cos \ \varphi \ d\varphi = \int_0^{\varphi_h} < T_e > \cos \ \varphi \ d\varphi \ . \tag{11}$$

We can now solve the problem, in principle. One way of visualizing the solution is graphically, as shown Fig. 4. Held & Hou used a thermal equilibrium profile such that $Z(z) = 1$ and

$$\Theta(\varphi) = -\Delta(\sin^2 \varphi - 1/3) \ . \tag{12}$$

Then $\langle T_e \rangle$ goes as φ^2 near the equator, while, from (10), $\langle T_m \rangle$ goes as φ^4. These different shapes make it clear (see Fig. 4) that one can find an "equal area" solution that has

$$\int_0^{\varphi_h} \left[\langle T_m \rangle - \langle T_e \rangle \right] \ \cos \varphi \ d\varphi = \int_0^{\varphi_h} \left[\langle T_m \rangle - \langle T_e \rangle \right] \ d(\sin \varphi) = 0$$

(so that this states that the net area between the curves, plotted against $\sin \varphi$, must vanish) and has T continuous across $\varphi = \varphi_h$.

One can actually solve the problem semi-analytically. The interface between the two regions is located where $\sin \varphi_h = Y$, where

$$\frac{1}{3}(4R - 1)Y^3 - \frac{Y^5}{1 - Y^2} - Y + \frac{1}{2} \ln \left(\frac{1 + Y}{1 - Y} \right) = 0 \ ,$$

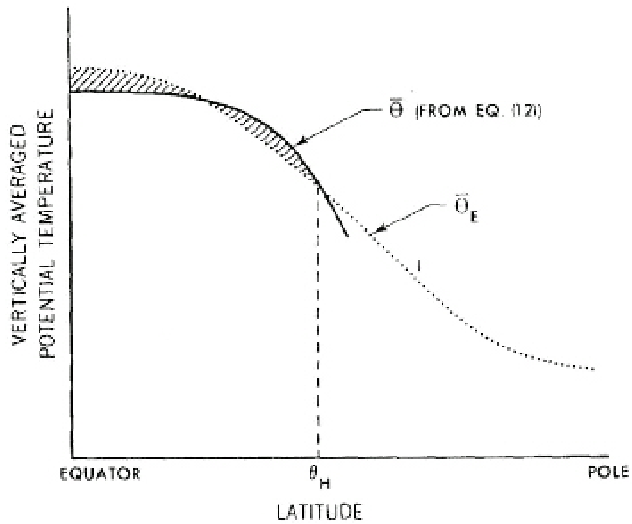


FIG. 1. The equal-area geometric construction equivalent to the argument of Section 4a. The two shaded areas are equal.

Figure 4: Solid curve is $\langle T_m(\varphi)\rangle$ from (10). Dashed curve is $\langle T_e(\varphi)\rangle$. {Fig 1. of Held & Hou. }

with

$$R = \frac{gH\Delta}{\Omega^2 a^2} \ .$$

For weak forcing, $R << 1$, the solution reduces to

$$\sin\varphi_h \cong Y \cong (\frac{5R}{3})^{\frac{1}{2}}$$

—so the width of "Hadley cell" goes as square root of equilibrium temperature contrast (*for this choice of forcing*). {Held and Hou also show the exact solution.}

Held & Hou did some numerical calculations for a series of values for ν—some of which are shown in Fig. 5. For small ν, the numerical solution asymptotes to something like what the inviscid theory predicts. [But not quite; the numerical solution goes inertially unstable, producing an unsteady state, for very small ν.] Fig. 6 shows how well $u(\varphi, H) \rightarrow u_m$ as $\nu \rightarrow 0$.

Note:

1. For small R, a small φ_h approximation $\sin\varphi \simeq \varphi$ is good (Held and Hou illustrate this.)

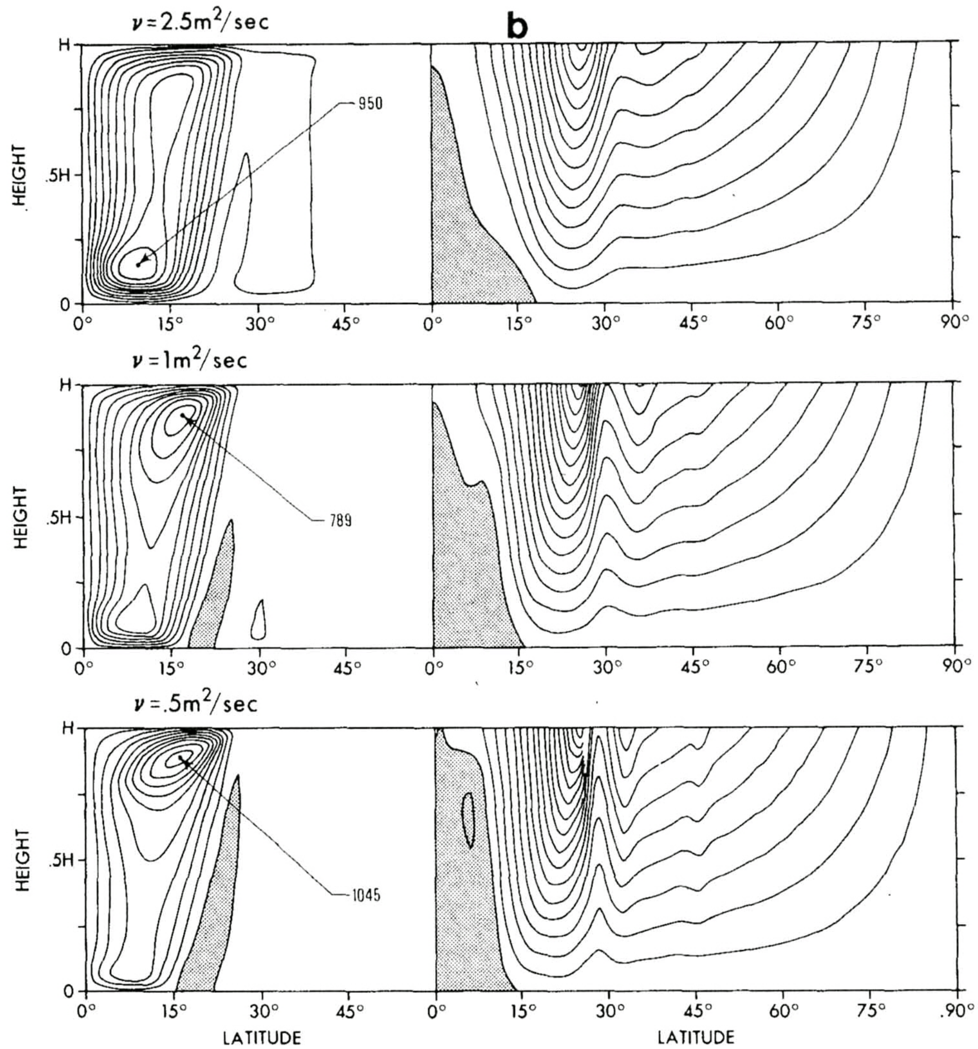


FIG. 4b. Calculated meridional streamfunctions and zonal wind fields as described in Fig. 4a. The shaded region in the ψ field corresponds to a Ferrel cell, $\psi < 0$.

Figure 5: {Held and Hou, Fig 4b.}

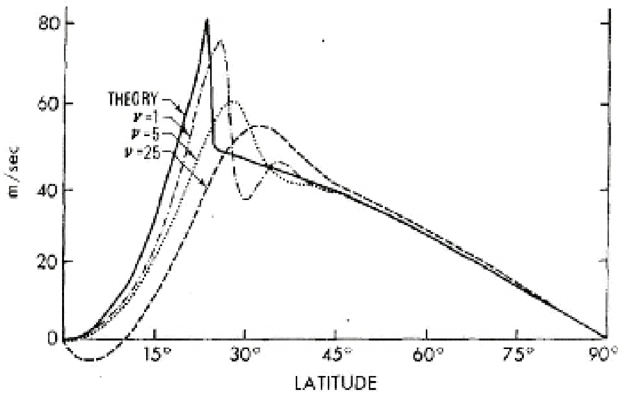


FIG. 5. Zonal winds at $z = H$ in the standard case for three values of ν, compared with the theoretical prediction for $\nu \to 0$.

Figure 6: {Held and Hou, Fig. 5}

2. The theory predicts that T becomes very flat within the tropical AMC regime. The AMC solution has

$$\langle T_m \rangle (\varphi) - \langle T_m \rangle (0) = -\frac{\Omega^2 a^2}{2gH} T_0 \frac{\sin^4 \varphi}{\cos^2 \varphi} \cong -\frac{\Omega^2 a^2}{2gH} T_0 \ \varphi^4 \ ;$$

cf.

$$\langle T_e \rangle (\varphi) - \langle T_e \rangle (0) = -\Delta \ T_0 \ \sin^2 \varphi \cong -\Delta \ T_0 \ \varphi^2 \ .$$

—angular momentum conservation makes $\langle T \rangle$ flatter than $\langle T_e \rangle$. (Recall the observed climatology of T.)

3. The edge of the circulation cell slopes poleward with z: the subtropical front.

4. In the Hadley cell, angular momentum conservation is violated (flow across m contours) at low levels and also near the upper boundary. See Fig. 7: m is constant along the top boundary for the small ν case for $15^0 < \varphi < 25^0$, but $m \neq M_0$. Suggests angular momentum exchange across meridional streamlines.

5. The AMC solution has $u(0, z) = 0$ and $u(\varphi, H) = \Omega a \ \sin^2 \varphi / \cos \varphi$: no tropical easterlies aloft or on equator. Numerical solutions show weak easterlies at low levels, especially for larger ν. $E - W$ boundary slopes *equatorward* with z.

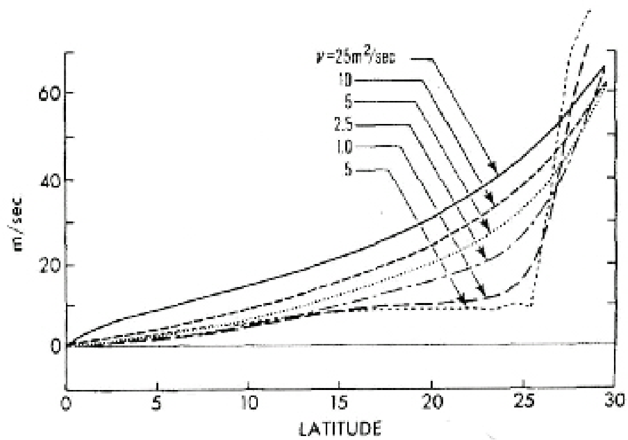


FIG. 6. $\Omega a \sin^2(\theta) - u \cos(\theta)$ at $z = H$ for various values of ν in the standard case.

Figure 7: {Fig. 6 of Held-Hou}

6. Because $\partial T/\partial\varphi$ is weak, then

$$wS \cong Q = -\alpha(T - T_e).$$

Note that this simply expresses a balance between radiative heating/cooling and adiabatic cooling/warming — where the air is cooler (warmer) than radiative equilibrium, it must be ascending (descending). Now, since $\langle T_m\rangle(\varphi_h) = \langle T_e\rangle(\varphi_h)$, for small R,

$$\langle T_e\rangle(\varphi_h) = -\Delta\, T_0 \left({\varphi_h}^2 - \frac{1}{3}\right) = \langle T_m\rangle(0) - \frac{\Omega^2 a^2}{2gH} T_0\, {\varphi_h}^4 = \langle T_m\rangle(0) - \frac{\Delta}{2R} T_0 \varphi_h^4 ,$$

and ${\varphi_h}^2 \cong 5R/3$ so

$$\frac{\langle T_m\rangle(0)}{T_0} = -\Delta\left(\frac{5}{3}R - \frac{1}{3}\right) + \frac{\Delta}{2R} \times \frac{25R^2}{9} = \Delta\left(\frac{1}{3} - \frac{5}{18}R\right) .$$

Therefore

$$\frac{\langle T_m\rangle(\varphi)}{T_0} = \frac{\langle T_m\rangle(0)}{T_0} - \frac{\Delta}{2R}\varphi^4 = .\Delta\left(\frac{1}{3} - \frac{5}{18}R\right) - \frac{\Delta}{2R}\varphi^4$$

and

$$\frac{[\langle T_m\rangle(\varphi) - \langle T_e(\varphi)\rangle]}{T_0} = -\frac{5}{18}R\Delta + \Delta\varphi^2 - \frac{\Delta}{2R}\varphi^4$$

If S independent of z, therefore,

$$\langle w \rangle (\varphi) = -\frac{\alpha}{S}[\langle T_m \rangle (\varphi) - \langle T_e \rangle (\varphi)] = \frac{\alpha \Delta}{2ST_0}(\frac{5}{9}R - 2\varphi^2 + \frac{1}{R}\varphi^4) \ .$$

If S also independent of Δ, $\langle w \rangle (0) \propto \Delta^2 (R \propto \Delta)$. $\langle w \rangle = 0$ at φ_0 where

$$\varphi_0 = \left(\frac{R}{3}\right)^{\frac{1}{2}} \ .$$

So the total upward mass flux is proportional to (for small R and therefore small φ) :

$$\int_0^{\varphi_0} w \ \cos\varphi \ d\varphi \simeq \int_0^{\varphi_0} w \ d\varphi \simeq \frac{5\alpha}{18ST_0} R \Delta \varphi_0 \propto \Delta^{5/2}.$$

A schematic summary of these results is shown in Fig. 8. This

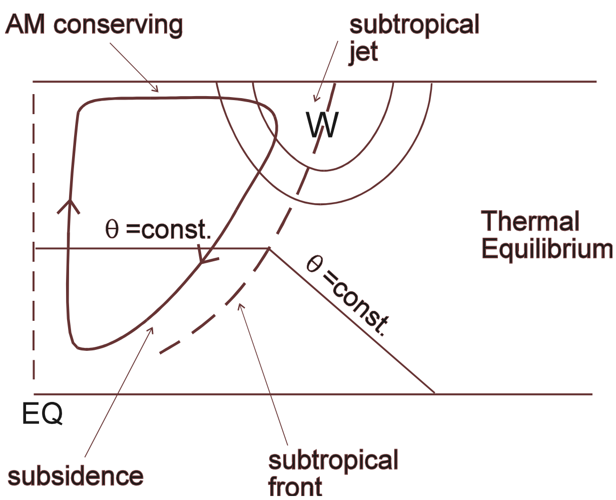


Figure 8: Schematic of theoretical Hadley cells.

schematic shows many features evident in observations: the subtropical jets, the weak temperature gradients between the jets, the meridional circulation, the dry subtropics consequent on mean subsidence, the easterly Trade winds in the tropics. But there are major discrepancies: the observed jets are weaker than this simple theory would predict, the observed surface extratropical westerlies are missing (or in the wrong place), and the high latitude atmosphere is not in radiative equilibrium (or it would be much colder than it is). So we are missing some important factors (eddies, of course). We are also missing, in this equinoctial or annually averaged view, asymmetry about the equator.

1.3 The asymmetric case [G]

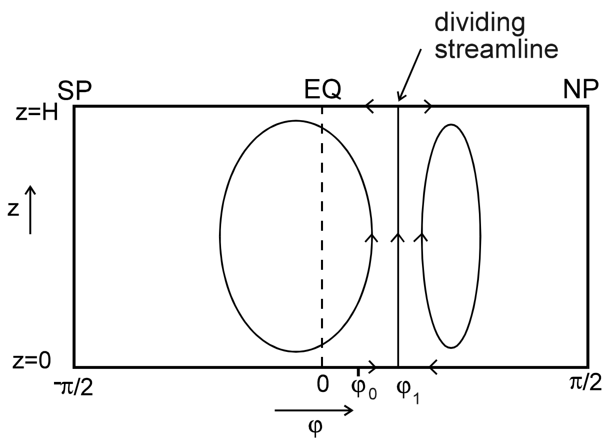
Lindzen and Hou (1988) took

$$T_e = T_{00}(z) + T_o\ \Theta(\varphi)Z(z)$$

with

$$\Theta(\varphi) = -\Delta[(\sin\varphi - \sin\varphi_0)^2 - \frac{1}{3}]\ . \qquad (13)$$

where φ_0 is the latitude of maximum $T_e(\varphi_0 = 0$ reduces to the Held & Hou case of eq. (12)). The theory goes basically the same way but with some differences:



1. cannot assume symmetry: $|\varphi_h^+| \neq |\varphi_h^-|$;

2. the dividing streamline (assumed vertical) is at φ_1, not at the equator; moreover, we cannot assume $\varphi_0 = \varphi_1$;

3. in the AMC cells, $M = \Omega a^2 \cos\varphi_1$, so

$$u_m(\varphi, H) = \Omega a \frac{(\cos^2\varphi_1 - \cos^2\varphi)}{\cos\ \varphi}$$

   As shown in Fig. 9, $u_m\,(\varphi, H)$ (and hence $\langle T_m\rangle$ also) is symmetric about *the equator*, not about φ_0 or φ_1, within the AMC cells (but these may not be symmetric in latitudinal extent). Note that there are easterlies in the equatorial upper troposphere for $\varphi_0 \neq 0$.

There are 4 unknowns —$\langle T\rangle\,(0)$, φ_1, φ_h^+, φ_h^-—and 4 constraints (2 matching conditions at each of 2 interfaces). Inviscid solutions for the last 3 look are shown in Fig. 10.

N.B.:

1. $\varphi_h^- > \varphi_h^+$: more spread into the winter hemisphere;

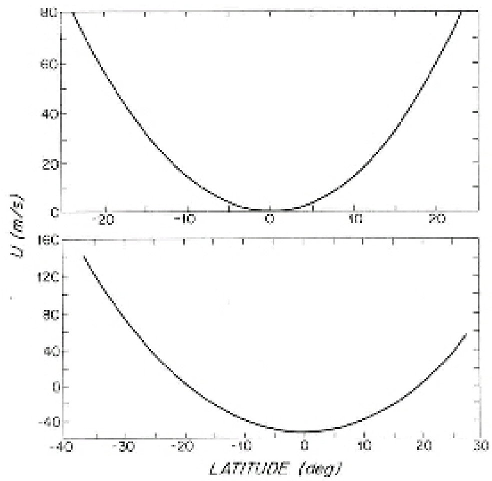


FIG. 6. $u(H, \phi)$ as a function of ϕ using the simple model. (a) $\phi_0 = 0$. (b) $\phi_0 = 6°$.

Figure 9: *[Lindzen & Hou, 1988]*

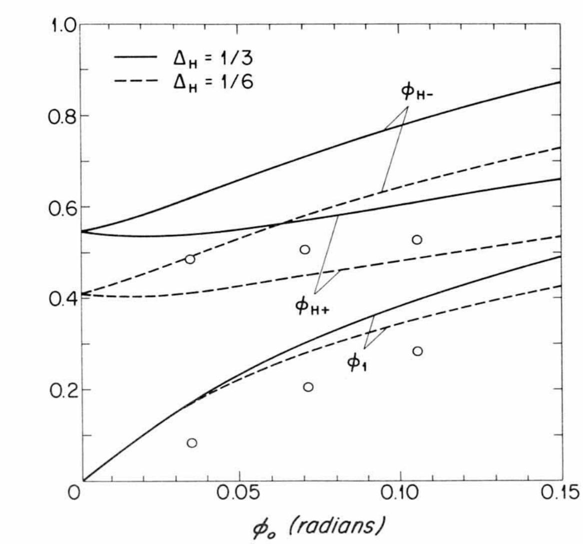


FIG. 4. ϕ_1, ϕ_{H^+} and ϕ_{H^-} as functions of ϕ_0 (see text for definitions). Open circles show results from numerical integration for ϕ_1 and ϕ_{H-1} when $\Delta_H = {}^1\!/_6$. (Note 1° of latitude $\approx$ 0.0175 radians.)

Figure 10: *[Lindzen & Hou, 1988]*

2. $\varphi_1 \gg \varphi_0$: the dividing streamline is much further poleward than the T_e maximum; the T_e maximum lies within the larger, cross-equatorial cell.

The temperature structure is

$$\langle T \rangle (\varphi) = \langle T \rangle (\varphi_1) - \frac{\Omega^2 a^2 T_0}{2gH} \frac{(\sin^2 \varphi - \sin^2 \varphi_1)^2}{\cos^2 \varphi}$$

which is shown in Fig. 11. For small angles,

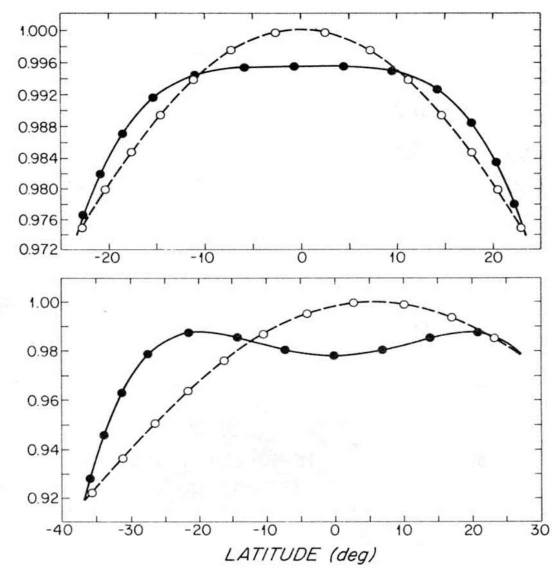


FIG. 5. $\bar{\theta}/\theta_0$ (solid line) and $\bar{\theta}_E/\theta_0$ (dashed line) as functions of ϕ using the simple model. (a) $\phi_0 = 0$. (b) $\phi_0 = 6°$.

Figure 11: *[Lindzen & Hou, 1988]*

$$\langle T \rangle (\varphi) \simeq \langle T \rangle (0) + \frac{\Omega a^2 T_0}{gH} ({\varphi_1}^2 \varphi^2 - \frac{1}{2} \varphi^4) \ ,$$

so

1. T has a local *minimum* at the equator for $\varphi_0 \neq 0$

2. T is maximum at $\varphi = \pm\varphi_1$ (for small φ_1).

Numerical solutions were obtained by Lindzen and Hou for $\nu = 0.5 m^2 s^{-1}$. The meridional streamfunction is shown in Fig. 12. for three

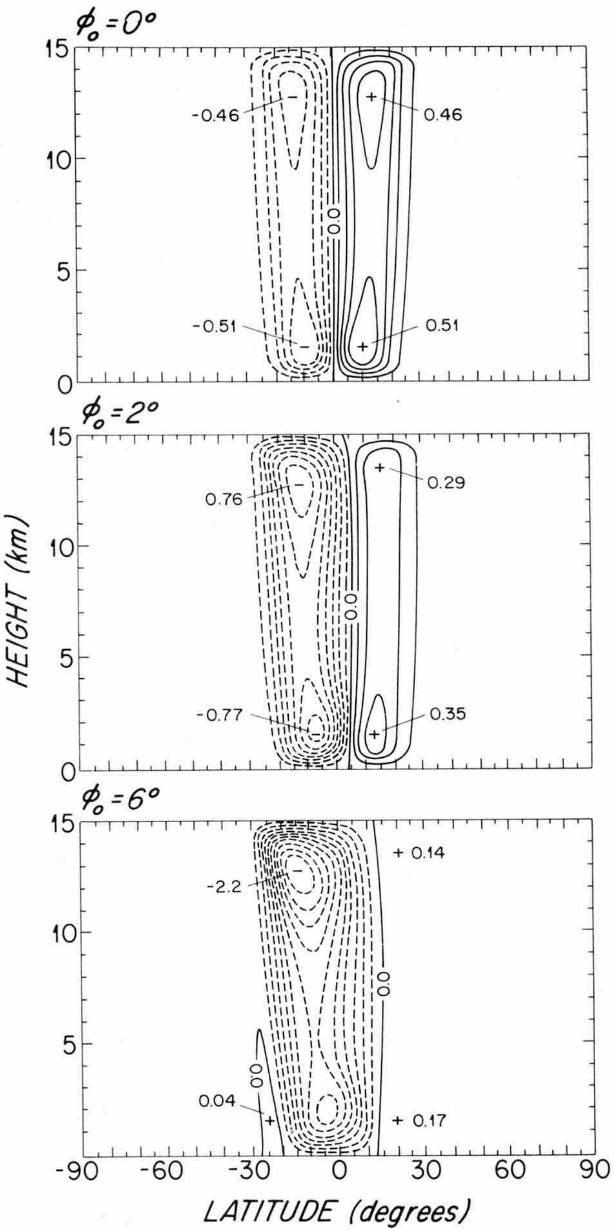


FIG. 9. Numerical model results for ψ for (a) $\phi_0 = 0°$, (b) $\phi_0 = 2°$, and (c) $\phi_0 = 6°$. Units are in 10^{10} kg s^{-1} and the contour interval is 0.1×10^{10} kg s^{-1} for (a) and (b); twice this value for (c).

Figure 12: *[Lindzen & Hou, 1988]*

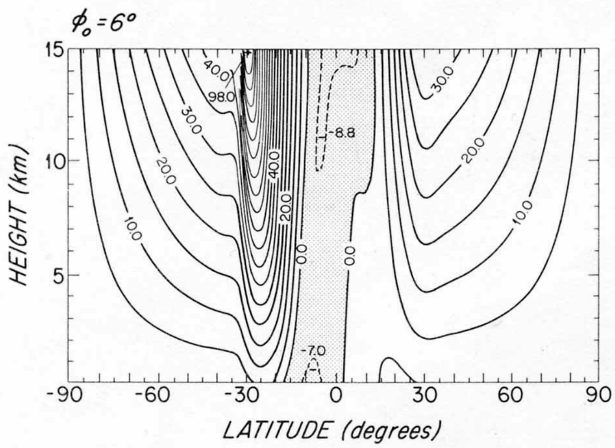


FIG. 10. The calculated zonal wind for $\phi_0 = 6°$ in m s^{-1}. Contour interval is 5 m s^{-1}.

Figure 13: *[Lindzen & Hou, 1988]*

values of φ_0. Note the strong asymmetry with even a small displacement φ_0 of the T_e maximum off the equator. The zonal wind distribution for $\varphi_0 = 6^o$ is shown in Fig. 13.

Note:

1. The jet in winter hemisphere is much stronger than that in summer.

2. Generally, the inviscid theory is not not too bad qualitatively (*e.g.* location of jets; departures from symmetry about the equator; weak equatorial easterlies) but poor quantitatively (jets much too strong). So the assumption of AM conservation does not look too good.

The corresponding temperature structure is shown in Fig. 14. Note especially the flat isotherms in the tropics, between the jets, and the strong baroclinic zone in the winter hemisphere. Apart from the exteme magnitude of the jets, note that the surface westerlies are located in the subtropics, beneath the subtropical jets, and not in middle latitudes.

Overall, the theory and the model results suggest the schematic of Fig. 15.

While the strong asymmetry is indeed evident in the solstice seasons, the senstivity of the asymmetry to small asymmetry in the external forcing may not be as great in reality as in these simplified calculations suggest [Dima & Wallace, 2003; Walker & Schneider, 2005].

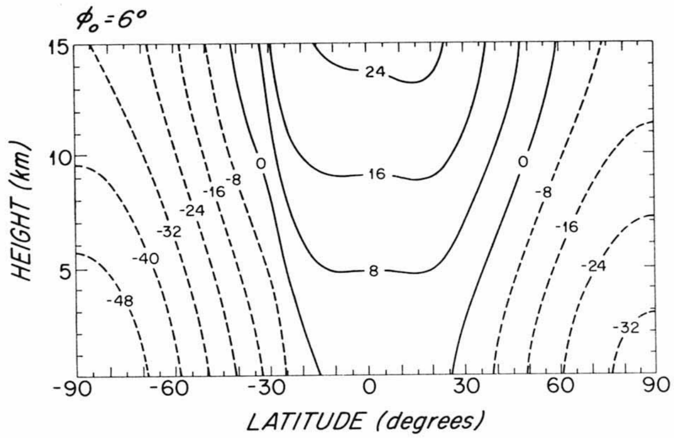


FIG. 12. The calculated temperature field for $\phi_0 = 6°$. The values correspond to departures from $\theta = 300$ K. The contour interval is 8 K.

Figure 14: *[Lindzen & Hou, 1988]*

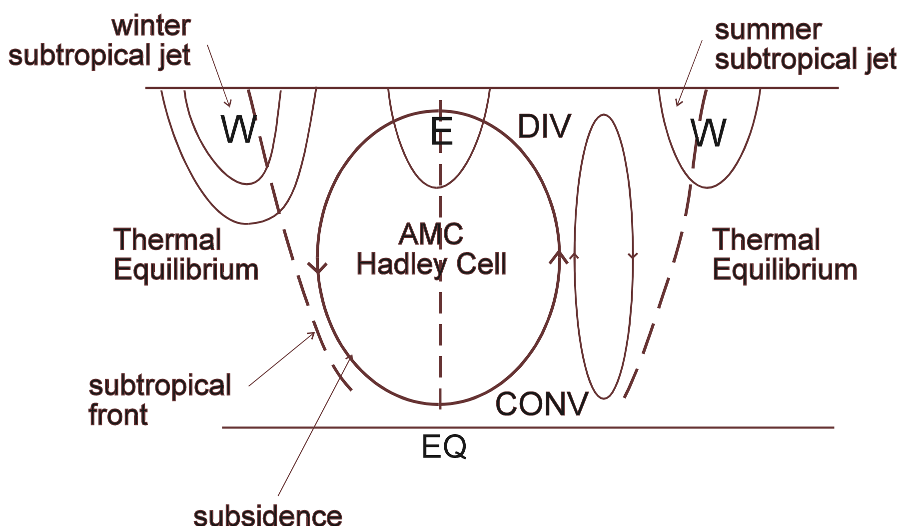


Figure 15: Schematic of the theoretical asymmetric Hadley cell.

Recall also that, at the edge of the theoretical inviscid Hadley cell, u need not be continuous; indeed, results show it is not (Fig. 6), at least for these simple forcings. Discontinuity of u implies a delta-function (and therefore an extremum) of vorticity, a state that might be expected to be unstable to 3D perturbations. Therefore, while these simple 2D models give a qualitatively satisfying picture of the tropical Hadley circulation, such a state may be impossible to achieve in practice.

In reality, of course, the flow is not zonally symmetric and this fact has a major impact on the actual tropical circulation, in two ways. First, transient synoptic eddies transport angular momentum out of the tropics and thereby cause violation of AM conservation. Second, the thermal driving of the circulation is strongly affected by zonal asymmetries in the lower boundary conditions: thus the circulation becomes regionalized on the continental scale, especially during northern summer when the zonally averaged circulation is dominated by the South Asia / Indian Ocean monsoon system.