Ruscher

① westerly in troposphere (increase with hght) and easterly in stratosphere ⟶ 3 questions.

② nuclear winter, global warming, acid rain, and the ozone hole ⟶ 2 questions

③ Jet stream dynamics. ⟶ 2 questions

④ QG dynamics ⟶ 5 questions.

⑤ Comma-shaped cloud (QG)

⑥ Vertical motion ⟶ 3 questions (including water vapor imagery method)

⑦ Petterssen development eq. ⟶ 3 questions.

⑧ Colder Tallahassee : PBL

⑨ incinerator (Air quality) : PBL

⑩ atmospheric chemistry studies given FSUGSM. : PBL

⑪ baroclinic/barotropic instability + atmosphere.

⑫ K.E. eq + Ri #

⑬ Basic PBL question.

westerly in troposphere + easterly in stratosphere

Synoptic (4501)

MET? : Ruscher (1 hour, 1994) *

- Discuss, starting with hydrostatics, the gas law, and Newton's second law, why there are westerlies in the middle latitudes in the troposphere. Also, discuss why the speed of the westerly current increases with height in the troposphere. Also, why do we find easterlies in the stratosphere?

Sol)

Start with

Hydrostatic eq.

$$\frac{\partial p}{\partial z} = -\rho g \Rightarrow \frac{\partial \Phi}{\partial p} = -\alpha \text{ in P coordinate } (d\Phi = g dz)$$

Eq of State

$$p = \rho RT$$

Newtons Second law

$$\underset{\sim}{F} = m\underset{\sim}{a}$$

→ applied to rotating spherical coordinate system of earth we obtain u, v, ω momentum eqs. The $u + v$ momentum eqs are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} - fv = -\frac{\partial \Phi}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial p} + fu = -\frac{\partial \Phi}{\partial y} + F_y$$

In these eqs we have neglected the curvature terms and those Coriolis terms involving ω and $\cos\phi$

Scale analysis shows that the large, synoptic scale motions to be in QG balance. We define the geostrophic wind $\underset{\sim}{V}_g$ as (in p coord.)

$$\boxed{\underset{\sim}{V}_g = \frac{1}{f}\mathbb{k} \times \nabla_p \Phi}$$

Of interest is the vertical shear in $\underset{\sim}{V}_g$

$$\frac{\partial \underset{\sim}{V}_g}{\partial \ln p} = \frac{1}{f}\mathbb{k} \times \nabla_p \left(\frac{\partial \Phi}{\partial \ln p}\right) \longrightarrow = p\frac{\partial \Phi}{\partial p} = p\left(-\frac{RT}{p}\right) = -RT$$

In a hydrostatic atmosphere $\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$ so since synoptic scale motions are largely hydrostatic we make this substitution and obtain

$$\boxed{\frac{\partial \underset{\sim}{V}_g}{\partial \ln p} = -\frac{R}{f}\mathbb{k} \times \nabla_p T} \quad : \text{Thermal wind relation}$$

In the troposphere we find max zonally averaged temperatures in the tropics and cooler temperature poleward. The meridional temperature gradient is larger (in absolute magnitude) in the winter hemisphere. To a first order approx. the zonally averaged winds and temperature satisfy the thermal wind relation. Thus in the vertical the strongest shear in $\underset{\sim}{V}_g$ occurs at the level where $\left|\frac{\partial T}{\partial y}\right|$ is largest. This tends to be near the tropopause. With frictional drag at the sfc and decreasing upward we expect the max $|\underset{\sim}{V}_g|$ in the upper troposphere. Consequence of eddy flux of eddy momentum in baroclinic waves (typical of mid-lats) supply westerly momentum which maintains a zonal jet in the time mean.

In the stratosphere there is a layer of heating due to O_3 ~1mb (50km) (O_3 absorbs incoming UV radiation). The reversal of the zonal mean temperature gradient in the vertical from cooling in the upper troposphere to warming in the stratosphere leads to a reversal in direction of the vertical shear in the geostrophic wind. Thus easterly

jets are found in the stratosphere in regions where $\frac{\partial T}{\partial |y|} > 0$ (in contrast to the upper troposphere where $\frac{\partial T}{\partial |y|} < 0$) N.H.

↑

Bad response!

→ Better response!

Consider the zonal component of the thermal wind relation $\frac{\partial \underset{\sim}{V}_g}{\partial \ln p} = -\frac{R}{f}\mathbb{k} \times \nabla_p T$

$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y}$ ⇒ vertical speed shear in u_g depends on meridional temperature gradient and Coriolis. Note that f changes sign between the hemisph

In the troposphere we have cold poles and a warm equator

Thus $\frac{\partial T}{\partial y} < 0$ in N.H but $\frac{\partial T}{\partial y} > 0$ in S.H (take positive y in direction of North pole

and so

$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y} < 0$ in N.H and $\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y} < 0$ (due to sign change in f)

In both hemispheres the shear profile for u_g is of the same sign. With $\frac{\partial u_g}{\partial \ln p} <$ we have u_g ↑ as P↓ or max westerly winds in the upper troposphere. This is what we observe in long term means ⇒ westerly jets in the upper tropospl of both hemispheres.

* In the upper stratosphere (above the influence of the troposphere → 40 to 50km or 1 mb) we find a temperat gradient predominantly directed from the cold winter pole to the warm summer pole. There is no equatorial max in temperature. The temp max is at the summer pole. Consider the NH winter, then in the stratosphere

$$\frac{\partial T}{\partial y} < 0 \text{ in NH and } \frac{\partial T}{\partial y} < 0 \text{ in SH}$$

So

$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y} < 0$ in NH but since f changes sign to $f < 0$ in SH

↓

u_g ↑ as P↓

↓

a westerly jet

$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y} > 0$ in SH

u_g ↑ as p↑ or u_g ↓ as P↓

⇓

an easterly jet

we expect a westerly jet in the winter hemisphere and an easterly jet in the summer hemisphere. This is, indeed, what we observe in the upper stratosphere westerly in troposphere

Synoptic (4501)

MET? : Ruscher (?, 1995)

- Using hydrostatics and basic arguments related to Newton's Laws of Motion, explain why winds aloft are westerly in the troposphere and why these westerly wind speeds increases with height in the troposphere. Include diagrams/sketches in your response.

Sol. for the previous + this question)

Hydrostatic eq $\frac{\partial p}{\partial z} = -\rho g \rightarrow \frac{\partial \Phi}{\partial p} = -\alpha = -\frac{1}{\rho}$ (as $d\Phi = g dz$)

Eq. of state (ideal gas law) : $p = \rho RT$ or $p\alpha = RT$

Newton's 2nd law : $\underset{\sim}{F} = m\underset{\sim}{a}$, when applied in spherical coords → eq of motion

$$\left.\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} = fv - \frac{\partial \Phi}{\partial x} + F_x \\ \frac{dv}{dt} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial p} = \underset{\text{Coliolis}}{\underset{\swarrow}{-fu}} - \underset{\text{PGF}}{\underset{\downarrow}{\frac{\partial \Phi}{\partial y}}} + \underset{\text{Friction}}{\underset{\downarrow}{F_y}} \end{aligned}\right\} \begin{array}{l}\text{Scaled eqs of motion} \\ \text{(neglected vert. coriolis term)}\end{array}$$

Scale analysis for mid-latitude synoptic systems → geostrophic approx (balance between Co. force + PGF) → $V_g = \frac{1}{f}\mathbb{k} \times \nabla_p \Phi$, a good approx for the wind there.

What about the vertical sheer? Differentrate V_g wrt p (pressure coord)

$$\rightarrow \frac{\partial V_g}{\partial p} = \frac{1}{f}\mathbb{k} \times \nabla_p \frac{\partial \Phi}{\partial p} = \frac{1}{f}\mathbb{k} \times \nabla_p(-\alpha) \text{ by hydrostatic eq.}$$

$\because -\alpha = -\frac{1}{\rho} = -\frac{RT}{p}$ by ideal gas law

$$\rightarrow \frac{\partial V_g}{\partial p} = -\frac{R}{f}\mathbb{k} \times \nabla_p \frac{T}{p}$$

$$\therefore \underline{\frac{\partial V_g}{\partial \ln p} = -\frac{R}{f}\mathbb{k} \times \nabla_p T} \text{ : Thermal wind eq}$$

The E-W component of this eq $\longrightarrow \underline{\frac{\partial u_g}{\partial \ln p} = +\frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p}$

So the vertical shear of the zonal wind is related to the Coriolis and the meridional temp gradient. We observe in the troposphere that the poles are cold w.r.t the tropics. So for our Coord. system where y increases from South → north, we get $\frac{\partial T}{\partial y} < 0$ in N.H. and $\frac{\partial T}{\partial y} > 0$ in S.H. As $f > 0$ in N.H. and $f < 0$ in S.H., we have vertical wind sheer of u_g as $\frac{\partial u_g}{\partial \ln p} < 0$ in both N.H + S.H. As u_g decreases as P increases

→ u increases with height in both N.H + S.H mid lats!

As u (defined positive for westerly) is small near sfc due to friction, we see wind is westerly and increases with height for mid-lat troposphere!

So what about the stratosphere? In the upper stratosphere (above the influence of troposphere) at P ~ 1mb (Z = 40-50km), we observe a temp. gradient directed from cold winter pole to warm summer pole, where there is no equatorial max as in the troposphere. Hence $\frac{\partial T}{\partial y} < 0$ $\left(\frac{\partial T}{\partial y} > 0\right)$ in the N.H summer (N.H. winter) for both hemisphere. Again as $f > 0$ for N.H + $f < 0$ for S.H, then you have

$$\frac{\partial u_g}{\partial \ln p} \begin{array}{l} < 0 \text{ for NH} \\ > 0 \text{ for SH} \end{array}\Big\rangle \text{ summer, and } \frac{\partial u_g}{\partial \ln p} \begin{array}{l} > 0 \text{ for NH} \\ < 0 \text{ for SH} \end{array}\Big\rangle \text{ winter.}$$

Assuming winds are light in this upper Strato. (B.C) and integrating down,

→ $\begin{array}{l}N \\ S\end{array}\left(\begin{array}{l}\text{westerly} \\ \text{easterly}\end{array}\right)$ max in the stratosphere in NH winter, and

$\begin{array}{l}N \\ S\end{array}\left(\begin{array}{l}\text{easterly} \\ \text{westerly}\end{array}\right)$ max in the stratosphere for NH summer

So, we expect an easterly max in the summer hemisphere, and a westerly max in the winter hemisphere for the stratosphere, which is observed!

Westerly in troposphere & easterly in stratosphere

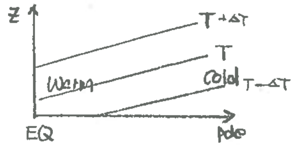
MET? (General question): Ruscher (1 hour, 1997) *
Synoptic (4501)

• Using arguments related to the general circulation of the atmosphere, hydrostatics, and radiation, explain (a) why we observe westerly winds in the middle latitudes of the troposphere; (b) why the westerlies increase with height; and (c) why we observe stratospheric easterlies.

Sol

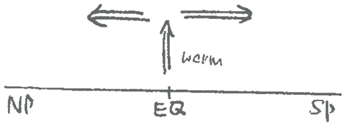
(a)

In the troposphere, especially in the lower level, the atmosphere temperature possesses such distribution in general that it is higher in tropics and lower in high latitudes



Such distribution is continuously maintained by radiation cooling in the high latitude and by heating in the tropics from solar radiation, convection and condensation

The warm air in the tropics will become lighter because by $p = \rho RT$, when $T\uparrow$, the density $\rho\downarrow$. Then the warm air must rise. At some altitude, the air must flow toward the poles by mass continuity requirement. This altitude is observed in upper level of troposphere.



when the flow moves toward the poles, the Coriolis effect will change its direction veer to the east (westerly)

By $\frac{\partial u}{\partial t} \sim f v$

Finally the wind will become totally westerly at middle latitudes. The air will then pile there and won't move further north (for $v \to 0$) and goes down to the ground. Thus, the westerly winds in the midlatitude are formed.

(b)

For synoptic scale motion, the atmosphere is quasi-geostrophic and quasi-hydrostatic. By the latter property, we can use p-coordinate system to describe the motion. So we have

$$f u \approx -\frac{\partial \Phi}{\partial y} \qquad (1)$$

with accuracy of $O(R_o)$, R_o : Rossby #

Φ : geopotential

u : westerly

f : coriolis parameter

Differenciate (1) with respect to p, we have

$$\frac{\partial u}{\partial p} = -\frac{1}{f}\frac{\partial}{\partial y}\frac{\partial \Phi}{\partial p} \qquad (2)$$

The hydrostatic gives

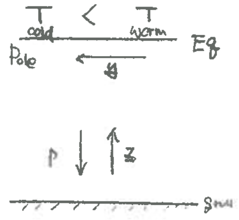
$$\frac{\partial \phi}{\partial p} = -\frac{1}{\rho} \qquad (3)$$

Substitute the state eq

$$p = \rho RT \qquad (4)$$

into (3) and then plug in result into (2), we have

$$\frac{\partial u}{\partial p} = \frac{1}{f}\frac{\partial}{\partial y}\left(\frac{RT}{p}\right) = \frac{R}{fp}\frac{\partial T}{\partial y} \qquad (5)$$



$\because \frac{\partial T}{\partial y} < 0$ at mid-latitude in the troposphere

$\therefore \frac{\partial u}{\partial p} < 0 \Rightarrow \frac{\partial u}{\partial z} > 0$

So the westerly increase with height.

(c)

When it comes to the stratosphere, it is observed that

$$\frac{\partial T}{\partial y} > 0$$

in the middle latitudes in some seasons.

Thus $\frac{\partial u}{\partial p} > 0 \Rightarrow \frac{\partial u}{\partial z} < 0$

So that at some altitude in the stratosphere we may expect $u < 0$. i.e. easterlies.

general

MET? : Ruscher (1 hour, 1994)

- Discuss each of the following controversial ideas as related to the notion of antropogenic factors influencing atmospheric changes. You do not have to present your own point of view, rather, I am interested here in seeing what the scientific arguments are related to each topic:

a) nuclear winter
b) global warming *
c) acid rain
d) the ozone hole *

a) Nuclear Winter

Numerous studies indicate that a nuclear war involving hundreds or thousands of nuclear detonations would drastically modify the earth's climate. Researchers assume that a nuclear war would cause an enormous pall of thick, sooty smoke from massive fires that would burn for days, even weeks, following an attack. The smoke would drift higher into the atmosphere where it would be caught in the upper level westeries and circle the midlatitude. Unlike soil dust which mainly scatters and reflects incoming solar radiation, soot particles readily absorb sunlight. Hence for several weeks after the war sunlight would virtually be unable to penetrate the smoke layer bringing darkness or, at least, twilight at midday.

Such a reduction in solar energy would cause surface air temperatures over land masses to drop below freezing, even during the summer, resulting in extensive damage to plants and crops and the death of millions of people. The dark, cold, and gloomy conditions that would be brought on by nuclear war is often refered to as nuclear winter.

As the lower troposphere cools, the solar energy absorbed by the smoke particles in the upper troposphere would cause this region to warm. The end result would be a strong, stable temperature inversion extending from the sfc up into the higher atmosphere. A strong inversion would lead to a number of adverse effects such as suppressing convection, altering precipitation processes, and causing major changes in the general wind patterns.

The heating of the upper part of the smoke cloud would cause it to rise upward into the stratosphere where it would then drift equatorward. Thus, about 1/3 of the smoke would remain in the atmosphere for a year or longer. The other 2/3 would be washed out in a month or two by precipitation. This smoke lofting, combined with persisting sea ice formed by the initial cooling would produce climatic change that would remain for several years.

Oxides of nitrogen produced by the nuclear explosions would be lofted into the stratosphere with the smoke where they would destroy ozone. The reduction of O_3 would allow excessive UV radiation to reach the sfc where it would have an adverse effect on plants, animals and humans.

Most research on this topic agrees with the above outcome. Observations of forest fires show a lowering of temps under the smoke. A three-year study (SCOPE) conducted by international scientists details the climatic, environmental, and agricultural effects of nuclear war.

b) Global warming

CO_2 is a trace gas which strongly absorbs LW radiation emitted by the earth and so plays a major role in warming the lower atmosphere. Similarly other trace gases such as methane (CH_4), nitrous oxide (N_2O), and CFCs all readily absorb IR radiation. The ability of these trace gases to absorb IR radiation and their role in warming the earth's atmosphere has led to the popular name "greenhouse" gases when referring to these chemical species. To be precise the way in which these gases lead to warming the atmosphere is not the same mechanism which is at work in a greenhouse. Nevertheless the name persists.

Over the past century the concentration of these gases has been steadily increasing. This coupled with their ability to absorb outgoing terrestrial radiat has led to much discussion/research about warming of the global atmosphere. Observations suggest that the surface of the earth has warmed by about $\frac{1}{3} \rightarrow \frac{1}{2}$°C since the start of this century. However there are uncertainties in the temperature record.

- movement of observing stations.
- new instruments + observing techniques
- increased urbanization → observations made in urban areas are anomalously high due to the urban heat island effect.

Is this warming due to anthropogenic causes? Is this warming merely a reflecti of natural variability in the earth climate system? Other data records (precip, satellite measurement) show no significant trend on the data over the recent past. This uncertainty is reflected in modeling studies.

Oceans + clouds appear to have an important role in the earth climate syste Our understanding of the complex interactions between the ocean + atmosphere hamper definitive answers to questions of global warming. Oceans have a huge heat capacity and thus they may slow the rate of any possible warming/cooling of the global atmosphere. The oceans are huge storehouses for CO_2. Photoplankton extract CO_2 from the atmosphere during photosynthesis and store some of it below the ocean sfc when they die.

→ a warmer earth might imply more photoplanktons ∴ reduce CO_2
→ a warmer ocean might increase atmospheric CO_2 since a warmer ocean can't hold as much CO_2 as a cooler one.

Clouds greatly complicate matters. How the climate responds to changes in global cloudiness depends on the types of clouds which form, their physical properties.

→ high thin cirroform clouds promote a net warming

→ low stratiform clouds promote a net cooling.

ERBE data show global cloudiness has a net cooling effect → possibly increased cloudiness brought on by global warming may experience negative feed back from increased cloudiness. Increased deep convection in a warmer atmosphere could actually lead to a warmer + drier atmosphere. The removal of atmospheric water vapor by deep convection could have a negative feedback on global warming (water vapor is a significant greenhouse gas).

Possible consequencies of global warming include the following.

- weakening of and shift in position of jet stream and other global wind patterns. Due to a warmer surface, evaporations increases ⇒ greater worldwide precip but since wind patterns have change the distribution of precip. changes.
  ⇒ could have significant impact on agriculture
- rise in oceanic sea levels → coastal flooding, contamination of coastal freshwater supply (salt water intrusion)
- response of warming in poles is very complex. Warming could actually lead to an increase in snow/ice pack due to more snowfall
- impact on plant + animal species
  → possibly more plant growth in warmer, moist climate but other dry weather species could die. Some animals might become extinct; others thrive.
- impact on upper atmosphere → warming in troposphere coupled with cooling in stratosphere (this is just theory – one of many)
  ↳ cooler stratosphere would reduce rate of O_3 destruction.

C) Acid rain

Air pollution emitted from industrial areas, especially products of combustion, such as oxides of sulfur and nitrogen, can be carried many km downwind. Either these particles and gases slowly settle to the ground in the dry form (dry deposition) or they are removed from the air during the formation of cloud particles and then carried to the ground in rain + snow (wet deposition). Acid rain and acid precip. are common terms used to describe wet deposition. Acid deposition encompasses both dry and wet acidic substances.

Emissions of sulfur dioxide (SO_2) and oxides of nitrogen settle on the local landscape where they transform into acids as they interact with water, especially during the formation of dew + frost. The remaining airborne particles may transform into tiny dilute drops of sulfuric acid (H_2SO_4) and nitric acid (HNO_3) during a complex series of chemical reactions involving sunlight, water vapor, and other gases. These acid particles may then slowly fall to earth, or they may adhere to cloud droplets or to fog droplets producing acid fog. They may even act as nuclei on which the cloud droplets begin to grow. When precipitation occurs in the cloud, it carries the acids to the ground. Because of this, precipitation is becoming extremely acidic in many parts of the world, especially downwind (up to 600 km) of major industrial areas. (Note that acid precip can form naturally ⇒ exposed beds of coal in northern Canada which catch fire due to natural causes put vast amount of SO_2 into air).

Acid precip can have a negative impact on plant + animal life when the soil/water is unable to neutralize the acid → ① "dead" lakes in eastern Canada + upper New England states, ② blight in trees of Germany has affected 1/3 of total trees, chemical erosion of manmade structures (monuments). While precip is naturally acidic (pH ~ 5.6) acid precip can have pH as low as 1.7 (LA region smog). Coal fired power plants in upper midwest US injects large amount of SO_2 into atmosphere → acid precip. downstream.

d) Ozone hole

Observations from satellites and balloon-borne instruments indicate that since the late 1970s, O_3 concentrations have diminished each year during Sep. + Oct in the Antarctic stratosphere. O_3 diminished by as mus as 40% in 1984 and shrank to the point of almost total depletion in certain regions of the lower stratosphere in 1987 and 1989. This decrease in stratospheric O_3 over Antarctica is known as the Ozone hole. In years of severe depletion (e.g. 1987, 1989) the O_3 hole covers almost twice the area of Antarctica. Data from high flying research aircraft showed that during 1987 significant amounts of O_3 were being destroyed outside the main hole. This result suggested that the area of O_3 loss could be much larger than thought. The same research showed O_3 destruction begins earlier in the year than expected.

Field experiments in 1986 + 1987 (NOZE-1,-2) gathered important data which help scientists understand aspects of the O_3 hole. The stratosphere above Antarctica has one of the world's highest O_3 concentrations. Most of this O_3 forms over the tropics and is brought to the Antarctic by stratospheric winds. During Sept + October (SH spring) a belt of stratospheric winds called the polar vortex encircles the Antarctic region near 60°S. These winds essentially isolate the cold stratospheric air from the warmer, mid-lat air. During the long dark Antarctic winter temperature inside the vortex can drop to −85°C. At these temps what little moisture there is in the air sublimates to form ice clouds → polar stratospheric clouds. These ice clouds are crucial in facilitating chemical interactions among N_2, H_2, and Cl_2 atoms. ⇒ the end result being destruction of O_3.

Data from NOZE-1,-2 point to the chlorine in CFCs as the main cause for the O_3 hole. But chemistry alone does not explain the O_3 hole. For exampl it does not explain the early decline in O_3 readings just as the O_3 hole is forming. Dynamics, not chemistry, may play a role here ⇒ mixing of stratospheric O_3-rich air with tropospheric O_3-poor air from below.

While airbone instruments indicated the formation a small O_3 hole over the polar arctic during 1989, careful analysis of satellite data showed no O_3 hole. Apparently several factors inhibit massive O_3 loss in the Arctic. The stratospheric circulation over the Arctic differs from that over Antarctica. The Arctic stratosphere is to warm for widespread development of clouds that help activate Cl molecules (for destruction of O_3)

general
MET? : Ruscher (?)

• Two very important issues facing humankind today are commonly referred to as "global warming" and "the ozone hole". The popular press often confuses these issues or simply states the principles involved incorrectly. Based upon your knowledge of basic meteorology and atmospheric radiation, discuss these two issues separately. In particular, discuss how global warming might lead to sea level change. Also discuss the the proposed relationship between the ozone hole and skin cancer. Feel free to include sketches in your response.

look at the previous answers.?

Solution for the previous question

(a) Nuclear winter

Basically, the process procedes as follows.
① Massive nuclear exchange ⟶ ② lots of debris lifted high in the atmosphere, as well as lots of fires ⟶ ③ Soot + smoke, which is also lifted into upper troposphere ⟶ ④ Heavy smoke/soot in upper troposphere ⟶ ⑤ absorption of solar radiation ⟶ ⑥ Drastic drop in sfc tempertures, below freezing even in summer (hence, the name, "nuclear winter") ⟶ ⑦ Warm upper troposphere + cold sfc ⟶ deep inversion ⟶ ⑧ Suppressed convection + major changes in global circulation ⟶ ⑨ Soot reaching stratosphere would remain there for a long time (~year), while that in the troposphere would eventually fall out due to rain ⟶ ⑩ Sea ice would expand due to initial cooling. This along with soot in stratosphere ⟶ ⑪ Climatic changes which could persist for years.

(b) Global warming

Global temp. records have shown an increase in average global temps during this century. However, is this believable, as ① observing stations have moved (local effects), ② New instruments and observing techniques (data consistency?) ③ Increased urbanization near many observing sites ⟶ warm bias due to heat island?
So, is this antropogenic? Those whose say yes point to increase in greenhouse gases (CFCs, CH_4, H_2O, and especially CO_2). These gases are almost transparent to VIS solar radiation but highly absorb IR, which contributes to heating of atmosphere. Increase in these gases is almost certainly anthropogenic, but will it necessarily lead to long-term warming? Skeptics have many reasons to say "Who knows?"

① Recent warming preceding by "little ice age", so this period of warming could simply be the climatic response

② Earth/Ocean feedback not totally understood. Warmer oceans could ⟶ more plankton ⟶ more CO_2 absorption ⟶ less greenhouse gas ⟶ Cooling

③ Role of clouds not properly handled. Warmer temps ⟶ more clouds ⟶ increased albedo ⟶ net cooling. This feed back not dealt with in most models

(c) Acid Rain

Air pollution from industry (especially, combustion products, such as sulfur oxides + Nitrogen oxides) are carried downwind. They either simply settle out (dry deposition), or they are carried down by precip. (wet deposition). Interaction with H_2O, along with a complex series of chemical reactions, can produce sulfuric (H_2SO_4) and nitric (HNO_3) acids. ⟶ "acid rain"
Areas along the sfc not capable of naturally neutralizes this acid can experience build-up of acidity. This can lead to "dead" bodies of water, as well as damage to vegetation + soil.

(d) Ozone hole

Observations in the 70's and 80's discovered a depletion of O_3 over Antarctica in SH winter. The area of depletion covers an area over the pole ⟶ the named O_3 "hole". During SH summer, this area has the highest concentration of O_3.
O_3 formed in the tropics is transported there by stratospheric winds. However, in winter, a belt of winds called "polar vortex" essentially isolates the cold stratosphere over the south pole, during which time temps can drop to -85°C. What little moisture there is sublimates ⟶ ice clouds, or polar stratospheric clouds. These clouds are crucial in facilitating chemical interactions among N_2, H_2, and Cl_2 (antropogenic in nature) ⟶ destruction of O_3. Even though presence of Chlorine has been linked to anthropogenic factors, the nature + location of the hole is dynamic in nature. For instance, there is no O_3 hole in the tropics, mid-lats or even at the North Pole! The dynamics are different in the artic - also it's too warm ⟶ widespread dev. of polar stratospheric clouds deemed essential for O_3 depletion. Note: O_3 is destroyed all the time. However, it is also produced as well. The concern is the artificial presence of Chlorine (from CFCs) which if unchecked could upset this balance.

Jet stream dynamics

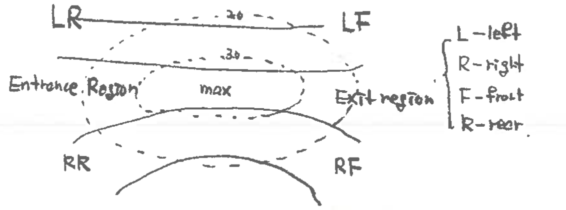
4501

MET? (General synoptic question): Ruscher (30 minutes) *

- Discuss how jet stream dynamics can play a role in intensifying and/or weakening surface cyclones. In particular, discuss the importance of the placement of a jet streak with respect to the surface cyclone and its development. Include a discussion of front- and rear-quadrant jet dynamics and secondary horizontal circulations and associated induced vertical motion patterns. Include sketches or diagrams in your response.

Sol)

An isotach maximum within a jet is called a jet streak. A jet streak normally propagates at a speed slower than the speed of the wind. Thus air parcels accelerate just upstream from the jet streak in the entrance region and decelerate just downstream from the jet streak in the exit region.



- Formation

we define the jetogenetical functions (assuming jet axis along x axis)

jet stream $\longrightarrow J = -f_0 \nabla_x^2 V_a - 2\beta \frac{\partial V}{\partial y} - \nabla_x^2 F_x \;;\; \nabla_x^2 = \frac{\partial^2(\,)}{\partial y^2} + \frac{\partial^2(\,)}{\partial z^2}$

embedded jet streak $\rightarrow J_s = -f_0 \nabla^2 V_a - 2\beta \frac{\partial V}{\partial y} - \nabla^2 F_x \;:\; \nabla^2 = \frac{\partial^2(\,)}{\partial x^2} + \frac{\partial^2(\,)}{\partial y^2} + \frac{\partial^2(\,)}{\partial z^2}$

$\beta = \frac{df}{dy}$ ⟶ beta effect

F_x ⟶ friction in zonal direction

$V_a = f(V - V_g) \equiv$ ageostrophic meridional wind speed

1st term

The jetogenetical function is positive whenever there is a local maximum in the cross jet component of the ageostrophic wind. (recall $-f_0 \nabla^2 V_a \propto f_0 V_a$). The ageostrophic wind is perpendicular and to the left of the parcel acceleration in the N.H.

↳ consider $V_a = \frac{1}{f} \mathbb{k} \times \left(\frac{\partial}{\partial t} + V_g \cdot \nabla\right) V_g$

V_a is perpendicular to $\frac{\partial V_g}{\partial t}$

Local parcel accelerations $\left(\left|\frac{\partial V_g}{\partial t}\right| > 0\right)$ lead to local wind maxima. If $-f_0 \nabla^2 V_a$ is positive somewhere, then by continuity there must be at least one branch of V_a in the opposite direction above or below. Thus vertical circulations are associated with the formation of Jet and jet streaks

2nd term

beta effect ⟶ usually negligible

3rd term $-\nabla^2 F_x \Rightarrow J \propto F_x$

Important for jets which form in the boundary layer such as the low level jet. Explanations for the llj include the following

- llj behaves like an inertial oscillation driven by the diurnal oscillation of friction.

[max cross isobar flow occurs during the day
[min 〃 〃 〃 〃 〃 〃 night → BL decouples from sfc layer and so frictional drag↓

and

[min total wind speeds occur at day
[max 〃 〃 〃 〃 〃 night

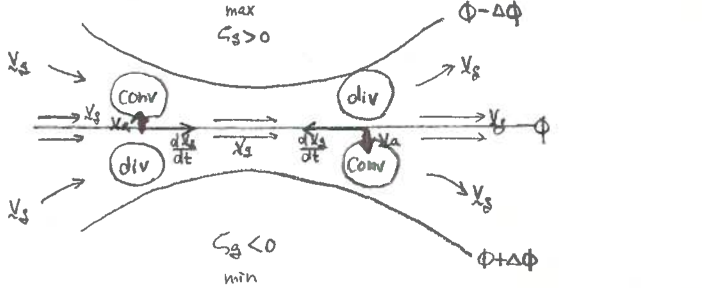
- llj responds to diurnal variations in sfc temperature via thermal wind considerations given $\nabla T \neq 0$
- For Great plains llj invoke western boundary intensification ideas used to explain Gulf Stream.

- Vertical motion

Recall QG momentum eq on an f plane

$$V_a = \frac{1}{f_0} \mathbb{k} \times \frac{D_g}{Dt} V_g$$

As an air parcel enters a jet streak $V_g \uparrow$. Therefore, as the parcel accelerate an ageostrophic wind blows perpendicular to the acceleration vector (left in NH) and toward lower heights. As the parcel exists the jet streak, $V_g \downarrow$ and an ageostrophic wind blows toward greater height. In this discussion we assume that accelerations due to curvature are small ⇒ a straight jet streak. As a result of these cross jet ageostrophic motions we have convergence in the left rear + righ front quadrants, divergence in the right rear + left front quadrants

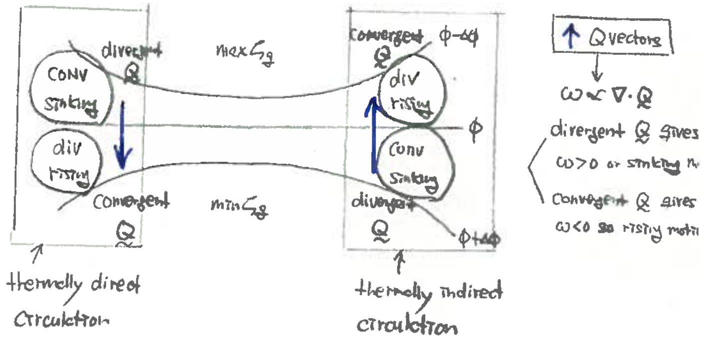


We find the same result if we apply the QG vort eq. $\frac{D_g}{Dt}(\zeta + f) = -f_0 (\nabla \cdot V$ to the above (and assume constant f)

If the jet is near the tropopause and we apply the typical tropopause BC on ω, namely $\omega = 0$, then according to continuity

$$\nabla \cdot V = -\frac{\partial \omega}{\partial p} \begin{cases} \text{conv} \Rightarrow -\frac{\partial \omega}{\partial p} < 0 \text{ or } \omega\uparrow \text{ with } p\uparrow \\ \text{div} \Rightarrow -\frac{\partial \omega}{\partial p} > 0 \text{ or } \omega\downarrow \text{ with } p\uparrow \end{cases}$$

there must be (rising motion below regions of divergence
(sinking motion 〃 〃 〃 convergence



- **Propagation**

The rate and direction of a jet streak can be determined from the field of height tendencies. Use the QG tendency eq

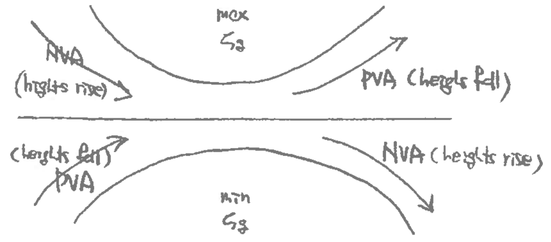
$$\left(\nabla^2+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi=-f_0 V_g\cdot\nabla(\zeta_g+f)-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-V_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right)$$

and neglect the temperature advection. We see

$\chi > 0$ where we have NVA
$\chi < 0$ where we have PVA

With regards to the jet streak we have NVA in the LR + RF quadrants ($\chi>0$), PVA in the LF + RR quadrants ($\chi<0$)

Assume advection of planetary vorticity is negligible



Based on the above, it follows that the jet streak will propagate downstream from west to east (left to right). Petterssen formula for the speed of a trough or ridge can be used to compute the actual QG speed since a jet streak is often defined by the junction of a trough and a ridge

- Coupling Jet streak to wind below

Underneath the rising branch of the transverse vertical circulation, at the sfc, there is convergence. Beneath the sinking branch is sfc divergence. According to the vorticity eq. convergence increases cyclonic vorticity whereas divergence leads to more anticyclonic vorticity ($\frac{d\zeta}{dt}=-(\zeta+f)\nabla\cdot\underline{V}$ + twisting/tilting term). Via continuity of mass, sfc pressure falls in the region of rising motion, $\left(\begin{smallmatrix}\text{upper div}\\ \text{lower con}\end{smallmatrix}\right)$, increasing cyclonic vorticity. In the region of increasing anticyclonic vorticity, surface pressure rises. Between the sfc pressure couplet will develop an ageostrophic flow from high to low pressure. This flow will be in the opposite direction of that at jet level.

In the exit region of a jet streak in westerly flow the sfc induced ageostropic flow is toward the pole while that aloft is toward the equator.

→ the poleward surface flow transports heat north and thereby reduces the meridional temperature gradient → this weakens the jet

→ the convergence of eddy flux of westerly momentum enhances the westerly jet

→ together the ageostrophic circulation associated with the jet tend to partly balance the influence of the transient eddy fluxes in order to maintain the mean thermal and momentum balance.

Low level warm air advection north destabilizes the atmosphere. Furthermore, if there is a supply of moisture equatorward of the jet, poleward advection of this moisture increases potential instability. The intersection of an upper level jet and low level warm/moisture ridge is a preferred location for severe weather

See Holton, 3rd ed., p348, Fig 10.16

Shapiro (1982) suggested that

① if the thermally indirect circulation associated with the exist region of an upper level jet streak and upper level front are situated directly over the thermally direct circulation associated with a low level front, then there is subsidence over the warm, moist air mass east of the front and convection is suppressed.

② If the circulation of the exit region of the jet streak moves farther east, it becomes coupled to the frontal circulation and the air mass is destabilized.

see Bluestein Vol II, Fig 2.105, p406

Sol)

The evolution and effects of jet streak circulations before and during explosive cyclogenesis.

– jet stream thinking –

Uccellin: the upper-tropospheric processes that lead to indirect ageostrop circulations and low-level jets affecting the east coast cyclogenesis process.

Shapiro: The formation of a jet streak can occur when the trough in the westerlies becomes in phase with a ridge at lower latitudes, to yield a flow configuration opposite of the high-over-low Rex block

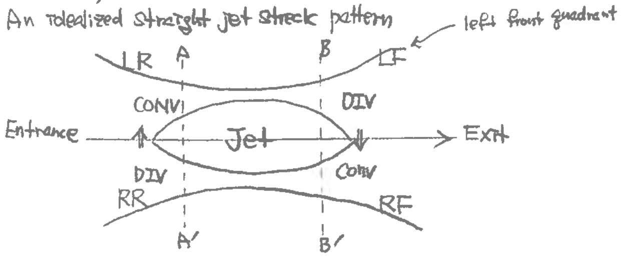
Palmer and Newton:

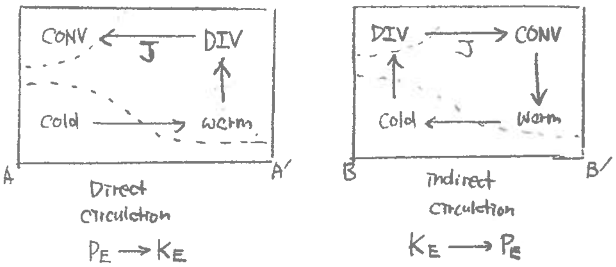
A jet streak – an isotach maximum imbedded within a jet associated with strong vertical + horizontal shear

In short, the evolution of jet streaks can be brought about by a pre-existing positive vorticity center due to curvature in the wind field, a tightening of the pressure gradient in a trough-ridge interaction, or the existence of an upper-level front and baroclinicity creating a strong thermal wind component on the warm side of the upper-level front.

- Methodology

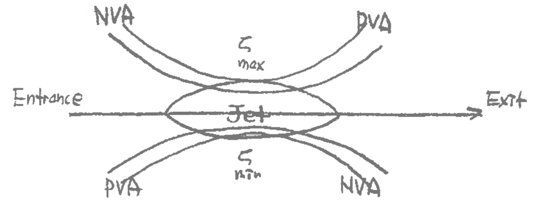
An idealized straight jet streak pattern





Vertical cross sections illustrating direct + indirect circulations in the entrance region (along A–A') and exit region (along B–B') of a jet streak (dotted lines denote isentropes). J marks upper-level jet location.

↓



* maximum (cyclonic) and minimum (anticyclonic) relative vorticity centers and associated advection patterns are associated with a straight jet streak

Due to cyclonic shear north of the isotach maximum, a positive vorticity center forms while a negative vorticity center forms south of the jet. PVA is found in the LF quadrant and NVA in the LR quadrant with a straight westerly flow superimposed over the vorticity pattern

Similarly, PVA + NVA are found in the RR and RF quadrent, respectively

From the vorticity eq.

$$\frac{D}{Dt}(\zeta+f) = -(\zeta+f)\nabla\cdot\underset{\sim}{V}$$

An air parcel moving through the RR quadrant experiences a decrease in relative vorticity. Since the change in ζ is much greater than that in f, the air parcel must experience divergence. This can apply to all quadrents for a divergence pattern.

From the divergence pattern, and by the QG omega eq, a secondary circulation pattern at the jet streak entrance and exit must develop to maintain the mass continuity in the atmosphere. A transverse or cross-stream ageostrophic wind component blows from the warm to cold air representing the upper banch of the direct transverse circulation. As air parcels accelerate into the jet in this region of confluence, the ageostrophic wind increases quiker than the actual wind (subgeostrophic). At the jet exit region the wind is supergeostrophic because the actual wind is greater than the geostrophic wind.

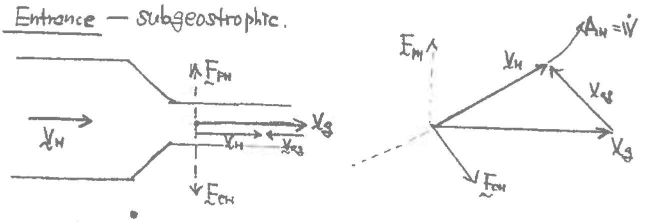
$$\underset{\sim}{V}_{ag} = \underset{\sim}{V}_H - \underset{\sim}{V}_g \quad ; \quad \underset{\sim}{V}_g = \frac{1}{f\rho}\mathbb{k}\times\nabla p$$

$$\underset{\sim}{F}_{CH} = f\mathbb{k}\times\underset{\sim}{V}_H \quad ; \quad \underset{\sim}{F}_{PH} = -\frac{1}{\rho}\nabla p$$

At the jet entrance, the ageostrophic wind component converts APE into KE for parcels accelerating into the jet.

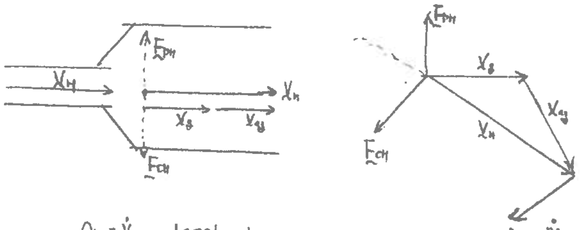
At the jet exit region, the ageostrophic component of the transverse indirect circulation is blowing from north to south towards lower heights

- Entrance — subgeostrophic.



$A_H = \dot{\underset{\sim}{V}}_H$ = acceleration : CON/DIV

- Exit — supergeostrophic



$A_H = \dot{V}_H$ = deceleration

Supergeostrophic :

The pressure gradient is now less then experienced in the jet core, therefore the PGF weakens while CF (slower to change) and horizontal wind component ($\underset{\sim}{V}_H$) remain strong. The net effect is an deceleration and turn to the right or southword. The secondary circulation are the restoring forces that return the air parcel to geostrophic balance in this region, converting KE to PE as the system decelerates. The lower branch of the indirect circulation from warm to cold air is the low-level jet.

Uccellin :

— the combined interaction between the polar and sub-tropical jet streaks as well as their influence on the low-level east coast jet at the time of explosive cyclogenesis

— The surface map showed a strong surface high below the entrance region of the polar jet, the explosive cyclogenesis occuring in the exit region of the sub-tropical jet and heavy precipitation in between.

Jet

Synoptic

MET? : Ruscher (1 hour) : **one figure!**

- Question

(1) Describe in very basic terms what is being displayed (i.e. fields and data plotted) on the chart below. (20%)

(2) In the region outlined by the heavy "box" there is a jet entrance region. In terms of simple kinematic principles, relate how the two terms which contribute to divergence interact in the exit region, including what sign they take on. Discuss your answer in terms of whether they reinforce each other or oppose each other. (30%)

(3) Discuss the secondary horizontal circulations and associated induced vertical motion patterns for this jet exit region and any reasons for the existence of secondary circulations in a jet exit region. Include additional sketches or diagrams in your response if you wish. (50%)

300 mb heights/isotachs 12Z SUN 8 NOV 92

1/

QG

4501

MET? (Synoptic question): Ruscher (1 hour, 1996)

- The two equations below form the basis of a particular set of equations often used in synoptic and dynamic meteorology. Discuss the complete system of equations, their appropriate use and limitations, and relevant assumptions for their application. Discuss the details of each of the two equations below for diagnosis of weather systems.

$$\underset{\textbf{A}}{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega} = \underset{\textbf{B}}{\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f_0\right)\right]} + \underset{\textbf{C}}{\frac{1}{\sigma}\nabla^2\left[\mathbf{V}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}$$

$$\underset{\textbf{A}}{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi} = \underset{\textbf{B}}{-f_0\mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f\right)} + \underset{\textbf{C}}{\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\mathbf{V}_g \cdot \nabla\frac{\partial\phi}{\partial p}\right)}$$

look at Holton's Ch6

1/1

QG

MET4501C: Ruscher (40 minutes) **

- Two fundamental relationships of quasi-geostrophic theory are shown in the following two equations

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega}_{\mathbf{A}} = \underbrace{\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f\right)\right]}_{\mathbf{B}} + \underbrace{\frac{1}{\sigma}\nabla^2\left[\mathbf{V}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}_{\mathbf{C}} \quad (1)$$

$\longrightarrow$ correct?

and

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi}_{\mathbf{A}} = \underbrace{-f_0\mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f\right)}_{\mathbf{B}} + \underbrace{\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\mathbf{V}_g \cdot \nabla\frac{\partial\phi}{\partial p}\right)}_{\mathbf{C}} \quad (2)$$

$\longrightarrow$ Correct?

(a) Name these two equations and give complete physical interpretations for each term, including the level at which they are valid. (b) What assumptions are made in order to develop this set of diagnostic equations? That is to say, can we apply this equation to any synoptic situation and does it cover the importance of any and all physical processes which could have an influence on the terms on the left had side? (c) Suppose we were interested in a situation of a developing wave cyclone. Sow how each of the terms in each equation would be related in order that maximum intensification is likely. Include sketches or diagrams in your response. (d) Describe what a Q vector is, how it relates to quasi-geostrophic theory, and what kind of Q vector pattern would be associated with cyclogenesis.

look at Holton's Ch6.

QG

4501

MET? : Ruscher (1 hour)

- The quasi-geostrophic system of equations include a static stability equation, a vorticity equation, an "omega" equation, and the following equation:

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi}_{\mathbf{A}} = \underbrace{-f_0\mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f\right)}_{\mathbf{B}} + \underbrace{\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\mathbf{V}_g \cdot \nabla\frac{\partial\phi}{\partial p}\right)}_{\mathbf{C}}$$

a. What is the name of this equation and at which vertical level is it generally applicable?

b. What is χ? Define it mathematically and tell what it means physically to have $\chi < 0$.

c. According to this equation, χ is related to two physical processes; what are they and which terms do they respond to?

d. What is required for terms **B** and **C** to be indicating $\chi < 0$? Discuss the processes and illustrate graphically the appropriate fields (sketch vertical profiles or map views).

e. What kinds of assumptions are invoked to develop this equation from the primitive equation set? Give two atmospheric situations where use of this equation would not seem appropriate based on these assumptions.

look at Holton's ch6

1,

QG

4501

MET? : Ruscher (?)

- The quasi-geostrophic system of equations include a static stability equation, a vorticity equation, an "omega" equation, and the tendency equation given below:

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi = -f_0\mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f\right) + \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\mathbf{V}_g \cdot \nabla\frac{\partial\phi}{\partial p}\right)$$

Read the entire question before you attempt to answer any part.

a) Define the concept of a secondary circulation within the framework of quasi-geostrophic theory. You may appeal to the basic assumptions under which quasi-geostrophic theory is derived. (10%)

b) Given the following diagram (next page) from Holton (An Introduction to Dynamic Meteorology, fig 7.11 and table) and the tendency equation above, fill in the table (next page) with a + or – depending upon whether the sign of the item under "physical parameter" is positive or negative. (70%)

c) For the vorticity tendency, which term in the quasi-geostrophic vorticity equation is dominating the sign of terms in locations A, B, and C on the figure and in the table? Indicate which term in the table is the vorticity tendency as part of your response. (20%)

look at Holton's ch 6

1/1

QG

MET? 4501 : Ruscher (1 hour)

- The quasi-geostrophic system of equations include a static stability equation, a vorticity equation, a tendency equation, and the following equation:

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega}_{\mathbf{A}} = \underbrace{\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f\right)\right]}_{\mathbf{B}} + \underbrace{\frac{1}{\sigma}\nabla^2\left[\mathbf{V}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}_{\mathbf{C}}$$

(1) What is the name of this equation and at what vertical levels is it applicable?

(2) According to this equation, vertical motion is related to two physical processes; what are they and which terms do they correspond to?

(3) When does term **A** represent *upward* vertical motion (assume a wave-like solution here)? [Hint: it might be easier to define upward motion first, using the symbols above.]

(4) What is required for terms **B** and **C** to be indicating *upward* vertical motion? Discuss the process and illustrate graphically the appropriate fields.

(5) What kinds of assumptions are invoked to develop this equation from the primitive equation set? Give two atmospheric situations where use of this equation would not seem appropriate based on these assumptions.

look at Holton's ch6

Note for QG theory + Q vectors. from Russ.

Two fundamental relationship of QG theory are shown below

ω-eq. →

$$\underset{A}{\left(\nabla^2+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)}\omega = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\underset{B}{\left[-V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)\right]}-\frac{1}{\sigma}\nabla^2\underset{C}{\left[-V_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}$$

tendency eq →

$$\underset{A}{\left(\nabla^2+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)}\chi = -f_0V_g\cdot\nabla\underset{B}{\left(\frac{1}{f_0}\nabla^2\phi+f\right)}-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\underset{C}{\left[-V_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}$$

$\chi = \frac{\partial\phi}{\partial t}$ $\quad$ ζ_g

Approx. made for these eqs

↳ assumptions → of these assumptions perhaps the neglect of diabatic effects and ageostropic advection are most open to criticism in particular situations

(a) hydrostatic balance in the vertical

(b) neglect of smaller (scale wise of synoptic scales) terms associated with spherical coordinates and rotating reference frame
↳ vertical coriolis term

(c) scale horizontal momentum eq so that terms $O(R_o^2)$ or smaller are neglected.

(d) assume constant f geostrophy so that $V_g = \frac{1}{f_0}\mathbb{k}\times\nabla\phi$

(e) apply eqs on mid-latitude beta plane : $f = f_0+\beta y$

∵ assume meridional scale is small
(compared to earth radius
$\frac{\beta L}{f_0} \sim O(R_o)$)

(f) approx $\frac{dV}{dt} = \frac{\partial V}{\partial t}+V\cdot\nabla V+\omega\frac{\partial V}{\partial p}$

as $\frac{dV_g}{dt} = \frac{\partial V_g}{\partial t}+V_g\cdot\nabla V_g$

but in thermodynamic eq retain $\omega\frac{\partial}{\partial p}(\;)$ since adiabatic heating/cooling is important

replace pgf $-\nabla\phi$ with $f_0\mathbb{k}\times V_g$

retain full wind in Coriolis term $f\mathbb{k}\times V = (f_0+\beta y)\mathbb{k}\times(V_g+V_a)$

combine $f\mathbb{k}\times V-\nabla\phi = f_0\cancel{\mathbb{k}\times V_g}-f_0\cancel{\mathbb{k}\times V_g}+f_0\mathbb{k}\times V_a + \beta y\mathbb{k}\times V_g+\underbrace{\beta y\mathbb{k}\times V_a}_{O(R_o^2)}) \simeq f_0\mathbb{k}\times V_a+\beta y\mathbb{k}\times V_g$

both $O(R_o)$ terms

(g) The following is not necessary to QG system but simplifies the resulting system of eqs. Expand temperature as

$$T_{tot}(x,y,p,t) = \underset{\text{basic state}}{T_0(p)}+\underset{\text{deviation from basic state}}{T(x,y,p,t)}$$

Then

$\left|\frac{dT_0}{dp}\right| \gg \left|\frac{\partial T}{\partial p}\right|$ and only T_0 needs to be included in the static

state $\boxed{\sigma = -\frac{RT_0}{p}\frac{d\ln\theta_0}{dp}}$

As a result of these assumptions, approximations, and simplifications we obtain the following QG momentum + thermodynamic energy eqs

$$\frac{dV_g}{dt} = -f_0\mathbb{k}\times V_a-\beta y\mathbb{k}\times V_g$$

$$\left(\frac{\partial}{\partial t}+V_g\cdot\nabla\right)T-\left(\frac{\sigma p}{R}\right)\omega = \frac{J}{C_p} \leftarrow \text{diabatic heating/cooling}$$

Since we assume hydrostatic atmosphere
$\frac{\partial\phi}{\partial p} = -\alpha = -\frac{RT}{p}$ can be used to replace T

If we now neglect diabatic effects (set J=0) we can obtain the hydrostat thermodynamic eq in the form

① $\frac{\partial\chi}{\partial p} = -V_g\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)-\sigma\omega \quad \left(\chi = \frac{\partial\phi}{\partial t}\right)$

and the vorticity eq.

② $\nabla^2\chi = -f_0V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)+f_0^2\frac{\partial\omega}{\partial p}$

QG system of equations
→ ω + φ are unknowns

Derivation of Tendency eq.

$\frac{f_0^2}{\sigma}① \Rightarrow \frac{f_0^2}{\sigma}\frac{\partial\chi}{\partial p} = -\frac{f_0^2}{\sigma}\left(-V_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right)-f_0^2\omega$

$\frac{\partial}{\partial p}(\;) \Rightarrow \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial\chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0^2}{\sigma}\left(-V_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]-f_0^2\frac{\partial\omega}{\partial p}\right.$

② $\Rightarrow \nabla^2\chi = -f_0V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)+f_0^2\frac{\partial\omega}{\partial p}$

$$\therefore \left[\nabla^2+\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)-\frac{\partial}{\partial p}\left(-\frac{f_0^2}{\sigma}V_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right)$$

Derivation of ω eq.

$-\frac{1}{\sigma}\nabla^2\frac{\partial\chi}{\partial p} = +\frac{1}{\sigma}\nabla^2\left[-V_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]+\nabla^2\omega$

$\frac{1}{\sigma}\frac{\partial}{\partial p}\nabla^2\chi = \frac{1}{\sigma}\frac{\partial}{\partial p}\left[-f_0V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)\right]+\frac{f_0^2}{\sigma}\frac{\partial^2\omega}{\partial p^2}$

$$\therefore \boxed{\left[\nabla^2+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)\right]-\frac{1}{\sigma}\nabla^2\left[-V_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}$$

- QG tendency eq

$$\underset{A}{\left[\nabla^2+\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]}\chi = -f_0V_g\cdot\nabla\underset{B}{\left(\frac{1}{f_0}\nabla^2\phi+f\right)}-\frac{\partial}{\partial p}\underset{C}{\left[-\frac{f_0^2}{\sigma}V_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}$$

Term A : Mathematical operator which spreads the response in the vertical so that forcing at one level influences other levels. For oscillating function the second order differential operator can be shown to return something proportional to $-\chi$. That is

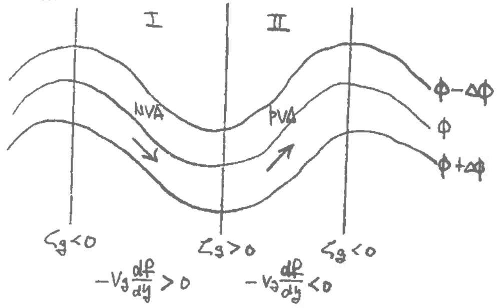
$$\left[\nabla^2+\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi \propto -\chi$$

Term B : $-f_0V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)$ ← vorticity advection.

This is generally the main forcing term in the upper troposphere. Break this up into two terms (neglect f_0)

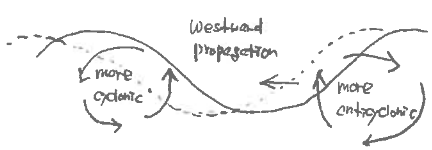
$$\underbrace{-V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi\right)}_{\substack{\text{geostrophic advection}\\\text{of relative vorticity}}}\;\underbrace{-V_g\frac{df}{dy}}_{\substack{\text{Beta term due to meridional advection of}\\\text{planetary vorticity}}}$$

For disturbances in the westerlies these terms tend to oppose each other as shown below



In region I upstream of the 500mb trough, the geostrophic wind has a north to south component. The Coriolis parameter increases poleward. Thus in region I, the northerly geostrophic wind component is advecting large planetary vorticity southward. That is, we have positive planetary vorticity. In region II downstream the 500mb trough, the situation is just opposite. With $V_g>0$, $-V_g\frac{df}{dy}<0 \Rightarrow$ negative planetary vorticity advection.

Planetary vorticity advection tends to move through ridge patterns to the west. ⇒ retrograde motion.



Contrast this to PVA in region II and NVA in region I which tend to move the ridge-trough pattern to the east. The actual displacement depends on which type of vorticity dominates

Suppose Φ is sinusoidal in $x + y \Rightarrow \Phi = \Phi(p) e^{ikx} e^{ily}$

$$k = \frac{2\pi}{L_x} , \quad l = \frac{2\pi}{L_y}$$

Then $\zeta_g = \frac{1}{f_0}\nabla^2\Phi = -\frac{(k^2+l^2)}{f_0}\Phi$

we see that as wavenumber $(k, l)\uparrow$ (wavelength $\downarrow$) $\zeta_g \uparrow$ so that relative vorticity advection dominates for short waves. The advection of planetary vorticity dominates for long wave

So, we see

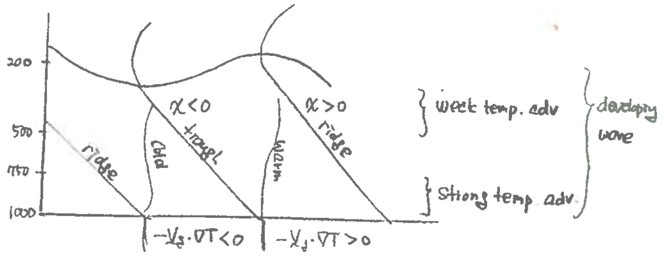
$$-\chi \propto -f_0 V_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)$$

$\Rightarrow$ PVA $\Rightarrow -\chi > 0$ so $\chi < 0 \Rightarrow$ height fall
NVA $\Rightarrow -\chi < 0$ so $\chi > 0 \Rightarrow$ height rise

It is important to note that vorticity advection can not change the strength of the disturbance. It merely serves to horizontally propagate the disturbance. Note also that vorticity advection is zero along the ridge & trough lines and tends to maximize midway between these lines

Term C $\quad -\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma} V_g \cdot \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$

This term represents differential temperature advection. It is the dominant term for amplification or decay of mid-lat synoptic systems. Temperature advection tends to be maximized near the sfc and much smaller (in terms of absolute magnitude) aloft. With regards to our 500mb baroclinic wave it is largest in the lower troposphere below the trough & ridge axes.



Beneath of 500mb ridge there is significant warm air advection near the sfc in association with the warm front. Cold air advection associated with the sfc cold front occurs beneath 500mb trough.

With regard to χ

$$-\chi \propto -\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma} V_g \cdot \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

temp adv (warm or cold) increases in direction of increasing p so sign of near sfc adv is sign of $\frac{\partial}{\partial p}(\ )$

or

$$\chi \propto \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma} V_g \cdot \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right] \begin{cases} > 0 & \text{for warm air advection beneath the ridge at 500mb} \\ < 0 & \text{for cold air advection beneath the trough at 500mb} \end{cases}$$

This makes sense when we think about it. Cold air advection reduces the thickness of the air in the column ⇒ 500mb height fall. Warm air advection into a column increase temp in the column. The thickness between pressure levels increase. 500mb heights rise

Summary

In the absence of diabatic heating the horizontal temperature advection must be nonzero in order that a mid-lat synoptic system intensify through baroclinic process. The temperature advection pattern described above indirectly implies a conversion of potential energy to K.E.

- QG ω-eq

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-V_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] - \frac{1}{\sigma}\nabla^2\left[-V_g \cdot \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

A B C

As in the tendency eq. the differential operator in term A spreads the response to a localized forcing (on the RHS). For oscillatory functions

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega \ (\text{or } +W)$$

So negative RHS forcing gives $-\omega < 0$ or $\omega > 0 \longrightarrow$ sinking motion
(positive " " " $-\omega > 0$ or $\omega < 0 \longrightarrow$ rising motion

* Advantages of QG-ω-eq for estimating ω

① Calculation of ω does not depend on ageostrophic wind (as in the kinematic met

② Only need data from single time (unlike vorticity and adiabatic methods)

③ QG ω eq ensures computed ω is consistent w/ QG vorticity & thermodyn fields $\rightarrow$ balance between mass in wind field in context of QG dynamic

* disadvantages

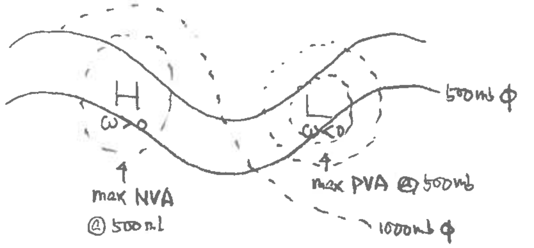
① large degree of cancellation between forcing terms.

② To obtain ω requires inversion of high order derivatives $\leftarrow$ accurately solving eq w/ noisy data is difficult

Term B : Differential vorticity advection $\quad -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-V_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right]$

using hydrostatic eq approximately write this as

$$\frac{f_0\alpha}{\sigma g}\frac{\partial}{\partial z}\left[\text{geostrophic advection of geostrophic absolute vorticity}\right]$$



Consider the above idealized developing baroclinic system. At 1000mb the vorticity advection must be very small at the center of the L and H whereas at 500mb it is very large in absolute magnitude. Thus, generally speaking, upper level vorticity advection > low level vorticity adv. and

$$-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-V_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] \simeq -\frac{f_0}{\sigma}\begin{bmatrix} \sim 0 - \text{NVA} \\ \sim 0 - \text{PVA} \end{bmatrix} \Big/ \Delta p$$

over H
over L

or

$$-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-V_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] \quad \begin{matrix} < 0 & \text{over sfc H} \\ > 0 & \text{" " L} \end{matrix}$$

In turn this implies

$$-\omega \begin{cases} < 0 & \text{over H} \\ > 0 & \text{over L} \end{cases}$$

or

$$\omega \begin{cases} > 0 \text{ over H} \longrightarrow \text{subsidence over sfc H.} \\ < 0 \text{ over L} \longrightarrow \text{rising motion over sfc L} \end{cases}$$

The pattern of vertical motion is just what is required to produce thickness tendencies in the 500–1000mb layer above the sfc high & low. Above the sfc low there is PVA associated w/ falling 500mb height. Hence the 500-1000 mb thickness is decreasing. Since horizontal temperature advection is small over the center of the sfc low, the only way to cool the atmosphere as required by the thickness tendency is by adiabatic cooling through the vertical motion field. Thus, the vertical motion field maintains a hydrostatic temperature field in the presence of differential vorticity advection. Without this compensating vertical motion either the vorticity changes at 500mb could not remain geostrophic or the temperature changes in the 500-1000 mb layer could not remain hydrostatic.

Term C :

$$-\omega \propto -\frac{1}{\sigma}\nabla^2\left[-V_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

$$-\omega \propto -\frac{1}{\sigma}V_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)$$

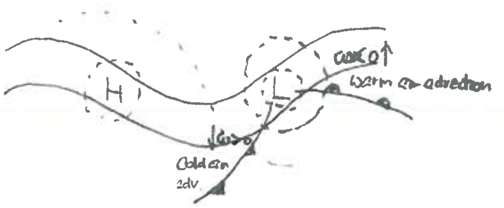
$\omega \propto -\left(\frac{1}{\sigma}\cdot\right.$ localized geostrophic temperature adv. $\left.\right)$

So, warm air advection (localized) implies $\omega<0 \longrightarrow$ rising motion

(localized cold air advection implies $\omega>0 \longrightarrow$ sinking .

Term C represents the effects of localized horizontal temperature advection on the vertical motion field.

Consider the idealized developing baroclinic wave below



Rising motion occurs to the east of the sfc L in the warm frontal zone. Sinking motion occurs to the SW of the sfc L behind the cold front. This vertical motion field is required to keep the upper level vorticity field geostrophic in the presence of height changes caused by the thermal advection. Geopotential height rises at the ridge and anti-cyclonic vorticity must increase if geostrophic balance is to be maintained. Since vorticity advection can not produce additional anticyclonic vorticity at the ridge, horizontal divergence is required to account for the negative vorticity tendency. Continuity of mass then requires that there be upward motion to replace the diverging air at upper level. By analogous arguments it can be shown that subsidence is required in the cold advection beneath the 500mb trough.

Summary

For synoptic scale motions where vorticity is constrained to be geostrophic and temperature is constrained to be hydrostatic, the vertical motion field is uniquely determined by the geopotential field. The vertical motion field is just what is required to ensure that changes in the vorticity will be geostrophic and changes in temperature will be hydrostatic.

- Idealized model of a barocrinic disturbance

Below are qualitative representations of the QG tendency & ω eqs

Tendency eq.

low level advection

geopotential $\binom{\text{fall}}{\text{rise}} \propto \binom{+}{-}$ vorticity advection $+ \binom{\text{Cold}}{\text{warm}}$ advection decreasing with height

Omega eq

upper level advection

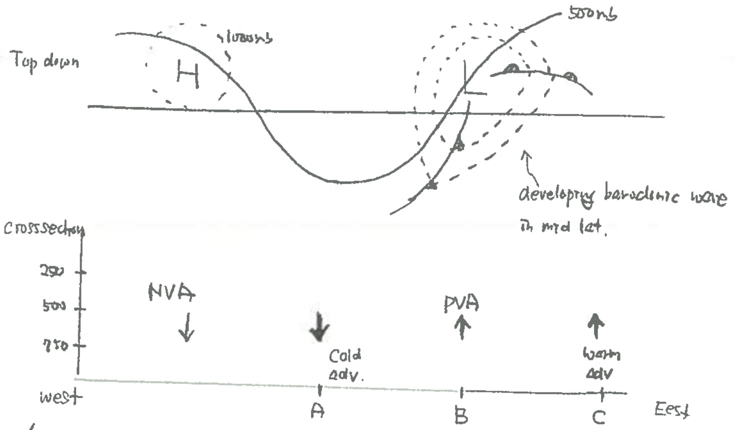
$\binom{\text{rising}}{\text{sinking}}$ motion $\propto$ rate of increase with height of $\binom{+}{-}$ vorticity advection

$+ \binom{\text{warm}}{\text{Cold}}$ advection

In QG dynamics the vertical motion & divergence fields act to keep the temperature changes hydrostatic and vorticity changes geostrophic in order to preserve thermal wind balance. We may regard the vertical and divergent ageostrophic motions as constituting a secondary circulation imposed by the simultaneous constraints of geostrophic & hydrostatic balance. ⟨It is interesting to note that the secondary circulation in a developing baroclinic system always acts to oppose the horizontal advection fields. Thus, the divergent motions tend partly to cancel the vorticity advection and the adiabatic temperature changes owing to vertical motion tend to cancel partly the thermal advection⟩

← self limiting mechanism

The diagram below summarizes these effects.



	A (500mb trough)	B (sfc L)	C (500mb ridge)
$\frac{\partial}{\partial t}$(1000-500 thickness)	negative but partially cancelled by adiabatic warming	negative due to adiabatic cooling	positive but partially cancelled by adiabatic cooling
W (500mb)	sinking	rising	rising
$\frac{\partial\Phi}{\partial t}$ (500mb)	negative due to differential thickness advection (cold air adv)	negative due to PVA	positive due to warm air adv.
$\frac{\partial\zeta_g}{\partial t}$ (1000mb)	negative due to sfc divergence	positive due to convergence	positive due to convergence
$\frac{\partial\zeta_g}{\partial t}$ (500mb)	positive due to convergence	positive due to PVA but partially cancelled by divergence	negative due to divergence

- Q-vectors

Recall the QG ω-eq.

$$\left(\nabla^2+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-V_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right] - \frac{1}{\sigma}\nabla^2\left[-V_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

$\binom{\text{rising}}{\text{sinking}}$ motion $\propto \frac{f_0}{\sigma}\frac{\partial}{\partial z}\left(\overset{\text{PVA}}{\underset{\text{NVA}}{\text{vorticity}}}\text{ advection}\right) + \frac{1}{\sigma}\left(\overset{\text{warm.}}{\underset{\text{cold.}}{\text{temperature}}}\text{ adv}\right)$

One problem in using the QG ω-eq. as given above to estimate ω is that there often is a large degree of cancellation between the two forcing terms on the RHS.

Another problem is that the above eq is not invariant under a Galilean transformation $\Rightarrow$ adding a constant zonal velocity to V_g changes the magnitude of two forcing terms without changing the net forcing of ω.

$\Downarrow$

Q vectors developed to address these shortcomings.

The ω eq. on a f-plane using Q vectors is

$$\left(\sigma\nabla^2 + f_0^2\frac{\partial^2}{\partial p^2}\right)\omega = -2\nabla\cdot\underset{\sim}{Q}$$

↳ so $+\omega \propto \nabla\cdot\underset{\sim}{Q}$

↳ (converging $\underset{\sim}{Q} \Rightarrow \omega<0$ ↑motion
diverging $\underset{\sim}{Q} \Rightarrow \omega>0$ ↓motion

where $\underset{\sim}{Q} = (Q_1, Q_2) = \left(-\frac{R}{p}\frac{\partial V_g}{\partial x}\cdot\nabla T,\ -\frac{R}{p}\frac{\partial V_g}{\partial y}\cdot\nabla T\right)$

To estimate Q on traditional wx maps we rewrite the Q eq as follows.

→ View motion in a cartesian coordinate system in which the x axis is parallel to the local isotherm with cold air on the left. In this case

$$\underset{\sim}{Q} = -\left(\frac{R}{p}\frac{\partial T}{\partial y}\right)\underbrace{\left(\frac{\partial V_g}{\partial x}\mathbb{i} - \frac{\partial u_g}{\partial y}\mathbb{j}\right)}$$

↳ apply the fact that the geostrophic wind is nondivergent on an f plane so

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$$

$$\underset{\sim}{Q} = -\frac{R}{p}\left|\frac{\partial T}{\partial y}\right|\left(\mathbb{k}\times\frac{\partial V_g}{\partial x}\right)$$

↑ minus comes from $-\left|\frac{\partial T}{\partial y}\right| = \frac{\partial T}{\partial y}$ since $\frac{\partial T}{\partial y} < 0$ by design.

From this form of the Q vector we see that we can estimate Q by

① evaluating the vectorial change of V_g along an isotherm with cold air on the left.

② rotating the resulting change vector by 90° CW.

③ multiplying the resulting vector by $\left|\frac{\partial T}{\partial y}\right|\cdot\frac{R}{p}$

Only require T + ϕ on same isobaric sfc.

Again, where (Q converge $\Rightarrow$ rising motion $\rightarrow \zeta$ in column ↑
Q diverge $\Rightarrow$ sinking motion $\rightarrow \zeta$ in column ↓

$\Rightarrow$ rising motion is associated with vorticity stretching in the column and so ζ↑. Sinking motion is associated with vorticity squashing and so ζ↓

$\Rightarrow$ look at my Holton notes !

1/

Comma-Shaped Cloud (QG)

Synoptic (4501)

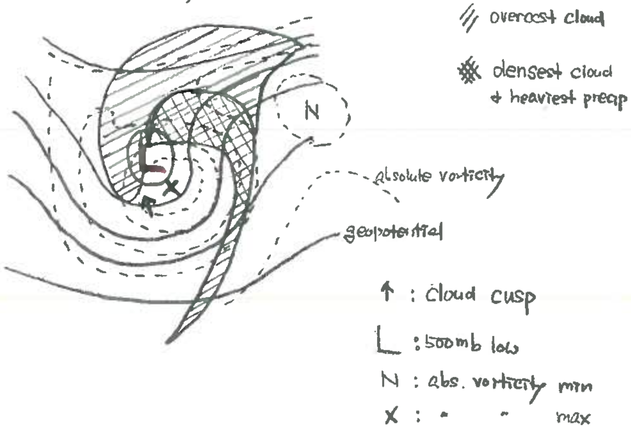
MET? : Ruscher (1 hour, 1994) *

• Provide the basic assumptions embodied within the quasi-geostrophic approximation. Next discuss, using QG-theory, why, when looking at a satellite picture of a mature extratropical cyclone, one sees a "comma"-shaped head, with a large expanse of cloudiness near and to the northeast of the low pressure system. In addition, cloudiness typically is confined to somewhat narrower bands to the southwest of the low center; explain why.

Sol) ⟶ description of comma cloud

· basic assumptions ⟶ look at other notes

Below is a schematic illustration of a typical comma cloud associated with a mature mid-lat cyclone



The comma cloud possesses

① a comma tail narrowing equatorward

② a comma head bulging toward the west.

During the stage of incipient sfc cyclone development, the cloud pattern tends to lie parallel to the upper wind flow in a band that is sometimes referred to as a baroclinic cloud leaf, because it is associated with strong tropospheric horizontal temperature gradients. In the developmental stage, the cloud narrows along its equatorward "tail" but expands on the poleward side, eventually developing a bulge westward on the poleward side of the sfc cyclone (the comma head). The comma head is often surrounded by a circular area of lower cloud, constituting the frictionally forced stratus cloud with tops below 700mb.

The clear zone between the comma tail + cloud to the west is the "dry tongue". Dry tongues are typical of strong cold air advection and PVA along its poleward edge. NVA is found equatorward and west of the 500mb $(\zeta+f)$ max. The later is situated equatorward and east of closed contours in the Φ field at 500mb. Consequently, air is descending over the equatorward side of the dry tongue and ascending on the poleward side of the vort. max. Because of the origins of this air entering the dry tongue (the high troposphere west of the trough), it is extremely dry at mid levels.

The figure on the above shows that closed Φ contours at 500mb are accompanied by the formation of a cusp of cloud at the end of the comma head (see the arrow). This cusp tends to be located very close to the 500mb low center and vorticity max. The sfc low, however, tends to be located at the point where the comma head joins the comma tail. Strong absolute vorticity advection over the comma head region, except along the western edge where there is only frictionally forced ascent. Clouds are generally much higher over the eastern part of the comma head, with the dividing line between very high + mid-level cloud tops lying close to the 500mb wind speed max which crosses the comma head. Warm air advection occurs over the comma head except where there is only frictionally driven low cloud.

Another aspect of cyclone development is the formation of a vorticity minimum along the poleward side of the comma head. There is a marked intrusion of lower absolute vorticity over the poleward edge of the comma head from the direction of the ridge. Accordingly, the poleward side of the comma head corresponds in this case to the limit of positive vorticity advection and to a region of developmental Φ rises in this area. Separate wind speed maxima tend to form in regions of enhanced Φ gradients between the max or min vorticity centers.

Sol)

First, recall basics of QG theory. In it we assume

(1) Hydrostatic balance, (2) Ideal gas law (3) $\zeta \simeq \zeta_g$

★ (4) $\underline{V} = \underline{V}_g$ in advection term (5) neglect vertical adv. + tilting/twist in vort. e

(6) assume atmosphere incompressible → use continuity eq

(7) β-plane approx ($f = f_0 + \beta y$) ⟶ more! (look at my Holton's note)

(★ note : $\underline{V}$ is <u>not</u> replaced by $\underline{V}_g$ in DIV term!)

We combine the TDE + vorticity eqs to eliminate $\chi = \frac{\partial \Phi}{\partial t}$ (geopotential tenden

we get

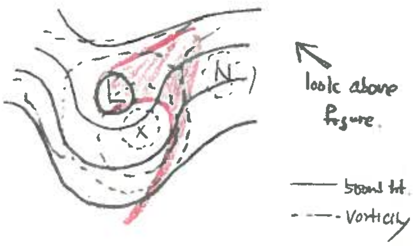
QG ω-eq ⟶ $\sim \nabla^2 \omega = \propto \frac{\partial}{\partial p}\left(\underline{V}_g \cdot \nabla(\zeta+f)\right) + \propto \nabla^2\left(\underline{V}_g \cdot \nabla T\right)$

assuming periodic sol. ⟶ $\sim -\omega = \sim \propto -\frac{\partial}{\partial p}(-\underline{V}_g \cdot \nabla(\zeta+f)) + \sim \propto (-\underline{V}_g \cdot \nabla T)$

→ $\begin{pmatrix}\text{rising}\\ \text{sinking}\end{pmatrix}$ motion ∝ increase with height of $\begin{pmatrix}\text{pos.}\\ \text{neg}\end{pmatrix}$ vort. adv. + $\begin{pmatrix}\text{warm}\\ \text{cold}\end{pmatrix}$ temp. adv

So what does a mature cyclone look like? Consider the following typical example

At 500mb: Note the vort. max (X) just SE of the 500mb low. The ridge accompany the trough → vort. min (N) ahead. Hence we see an area of <u>strong</u> PVA shaded in red, which according the QG theory should contribute to strong upward vert. motion. This area is in fact where the densest cloud + heaviest sustained is observed in a mature cyclone



4

The lack of cloudiness to the south of the vort. max is associated with strong cold temp. advection, which contributes to downward motion by QG theory. We notice a lack of cloudiness here; this area is often referred to as the "dry tongue", as the source of the air (upper troposphere west of trough) and descending → dry at mid levels. The thin area of clouds trailing to the south corr to the cold front, and is associated with convection-triggered by narrow frontal boundary. Note - Strong cold advection behind it → downward motion.

vert. motion

MET? (Synoptic question): Ruscher (1.5 hour, 1997) *

- Explain quantitatively and qualitatively the following methods of determining vertical motion in the atmosphere:
    - (a) the kinematic method
    - (b) the adiabatic method
    - (c) the Quasi-Geostrophic method
    - (d) the Q-vector method

Be sure to include in your response a discussion of advantages, disadvantages, assumptions, and appropriate uses.

look at Fuelberg's study note!

vertical-velocity

Q-vector

4501

MET? : Ruscher (1 hour) *

- There are many ways in which vertical motion may be diagnosed in the troposphere. Some of the methods include the (a) kinematic method, (b) the adiabatic/isentropic method(s), (c) the vorticity method, (d) the infrared satellite photograph method, and (e) use of quasi-geostrophic theory.

(Part I) Pick one of the methods from (a)-(d) and give a **full** description of the method and how it can be used to estimate/diagnose vertical motion. (30%)

(Part II) A convenient method for determining the quasi-geostrophic forcing effects on vertical motion is to use so-called Q-vectors, where Q is defined as follows:

$$\mathbf{Q} \equiv -\frac{R}{\sigma p}\begin{bmatrix} \frac{\partial \mathbf{V}_g}{\partial x} \cdot \nabla_p T \\ \frac{\partial \mathbf{V}_g}{\partial y} \cdot \nabla_p T \end{bmatrix} = \begin{bmatrix} Q_x \\ Q_y \end{bmatrix} \qquad (1)$$

Then the quasi-geostrophic omega equation becomes:

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -2\nabla_p \cdot Q - \frac{R}{\sigma p}\beta\frac{\partial T}{\partial x} \qquad (2)$$

Discuss each term in this equation and discuss how this equation can be used to diagnose vertical motion. (70%)

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look at Holtons ch6

* look at the fuelberg study note

Q-vectors

Recall QG ω-eq

$$\underbrace{\left(\nabla^2+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega}_{\sim -\omega} = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\mathbf{V}_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]}_{\text{vert. shear of vort. adv.}} \underbrace{-\frac{1}{\sigma}\nabla^2\left[-\mathbf{V}_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]}_{-\nabla^2\text{temp adv} \sim \text{temp. adv.}}$$

1) often large degree of cancellation between vort. adv. + temp. adv terms

2) Not invariant under a Galileen transform → adding a constant velocity

<u>changes</u> the magnitude of the terms, but does not change net forcing of ω!

If we combine terms, we get

$$\underbrace{\left(\nabla_p^2+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega}_{\sim -\omega} = \underbrace{-2\nabla\cdot\mathbf{Q}}_{\text{comb. of forcing}} \underbrace{-\frac{R}{\sigma p}\beta\frac{\partial T}{\partial x}}_{\text{adv of earth vort }(f)} \quad , \quad \underset{\sim}{Q} \rightarrow \text{defined above!}$$

If we neglect adv. of earth's vort., we get conv.(DIV) of Q → rising (sinking) motion! So rising (sinking) motion associated with vorticity stretching (squashing) in column, so $\zeta\uparrow$ ($\zeta\downarrow$)

The eastest way to evelucte this on WX maps is to transform local coord. system where $\underset{\sim}{X}$(ii) lies along isotherm with cold air to the left.

(as for V_T) $\rightarrow \frac{\partial T}{\partial x} \equiv 0 \longrightarrow \nabla_p T = \frac{\partial T}{\partial y}$ (Note as cold air to left, $\frac{\partial T}{\partial y} < 0$ always)

$$\rightarrow \overset{\checkmark}{\underset{\sim}{Q}} = -\frac{R}{\sigma p}\left|\frac{\partial T}{\partial y}\right| \mathbf{k} \times \overset{?}{\frac{\partial \mathbf{V}_g}{\partial x}} = \frac{R}{\sigma p}\left|\frac{\partial T}{\partial y}\right|\frac{\partial \mathbf{V}_g}{\partial x}$$

with direction $-\mathbf{k}\times\mathbf{V}_g$ → direction 90° to right of $\frac{\partial \mathbf{V}_g}{\partial x}$

Evaluate Q on either side (± increment in x direction) of a point.

If components along x are toward (away) from each other

→ conv (div) of Q → rising (sinking) motion.

1

Vertical motion — water vapor imagery.

4561

MET? : Ruscher (1 hour) *

- The diagnosis of vertical motion is one of the difficult problems which synoptic meteorologists must deal with in research and operational settings. This question deals with aspects of such diagnosis in the context of a very specific example. You must answer all three parts; weights are given for each part.

(a) [20%] One of the types of imagery available to meteorologists for the purpose of satellite interpretation is so-called "water vapor imagery". Discuss briefly the characteristics of the radiation involved, the meteorological variable and part of the atmosphere involved in the radiation, and the characteristics of the imagery (relationship of gray-scale to the weather element). Also, contrast water vapor imagery to visible and conventional infrared imagery in your answer.

(b) [60%] Water vapor imagery is widely used by meteorologists all over the world for various purposes. Some controversy about its use has been noted, however, regarding the inference of vertical motion just upstream of the mature middle-latitude cyclone; there is often a marked contrast in radiation intensity on the imagery in this region. Discuss how you would go about attempting to evaluate vertical motion (*without* the use of a numerical model) in this region of sharp gray-scale contrast (what kinds of analyses and calculations would be most useful?). In your answer, include some background on the problem (what are the expectations for vertical motion in this region based on observations and theory of synoptic scale cyclones?) and include at least one sketch.

(c) [20%] Discuss how a numerical model simulation might assist in the diagnosis of vertical motion upstream of the cyclone for individual cases of interest.

?

Pettersen development eq.

4501

MET? : Ruscher (1 hour)

- A prognostic equation often used in synoptic meteorology is given by:

$$\frac{\partial Z_0}{\partial t} = A_Z - \frac{R}{f}\nabla^2\left(\frac{g}{R}A_{\Delta Z} + S + H\right) - \mathbf{V}_0 \cdot \nabla Z_0 - Z(\nabla \cdot \mathbf{V}) \qquad (1)$$

small z

where

$$A_Z \equiv -\mathbf{V} \cdot \nabla Z$$

$$A_{\Delta Z} \equiv -\mathbf{V} \cdot \nabla(\Delta Z)$$

small z

Note. Z_o refers to ς+f, absolute vorticity!
(little z is still vert. coord.)

$$S \equiv \ln\left(\frac{p_0}{p}\right)\overline{\omega(\Gamma_a - \Gamma)}$$

$$H \equiv \ln\left(\frac{p_0}{p}\right)\frac{1}{c_p}\overline{\frac{dW}{dt}}$$

and a subscript 0 refers to the surface and non-subscripted terms refer to some upper level in the troposphere.

(1) What is the name of this equation? (5%) Pettersen development eq.

(2) How is cyclogenesis defined with this equation? (10%)

(3) At what level in the troposphere do we evaluate the terms on the right-hand side of Eq. 1? (10%)

(4) What is the form of the equation which is used most typically? That is, show which terms are normally important and discuss why one or more other terms are not evaluated routinely. (20%)

(5) Define and discuss each of the remaining terms, and give an illustration of how each term may contribute to cyclogenesis, including the stage or part of the life cycle of the cyclone at which each term becomes involved typically. Include sketches or descriptions of processes which would lead to cyclogenesis. (55%)

look at the attached note

Sol)

(1) Petterssen's development eq.

(2) A cyclone is considered to be developing when $\frac{\partial Z_o}{\partial t} > 0$

(3) At the level of non-divergence (LND), usually approx at 500mb.

(4) By eval. at the LND, we assume $\nabla \cdot \underline{V}$ is small so we drop the $Z(\nabla \cdot \underline{V})$ term (last term on RHS). Also as vort is small at sfc, we neglect vort. adv. at sfc $[-\underline{V}_o \cdot \nabla Z_o]$. Hence the common form of the eq. is

$$\frac{\partial Z_o}{\partial t} = A_Z - \frac{R}{f}\nabla^2\left(\frac{g}{R}A_{\Delta z} + S + H\right)$$

(5)

a) $A_Z = -\underline{V} \cdot \nabla Z \longrightarrow$ horizontal vorticity adv. at LND.

PVA (NVA) $\longrightarrow \frac{\partial Z_o}{\partial t} > 0\ (<0)$, which is just as one would expect from QG theory. Petterssen states this is the most important term for initial development. This is why forecasters often look for a shortwave (positive vorticity anomaly) migrating into the longer wave disturbance ⟶ development!

b) $A_{\Delta z} = -\underline{V} \cdot \nabla(\Delta z) \rightarrow$ horizontal thickness advection.

This term is most important for maintenance of system, where warm advection $\longrightarrow \frac{\partial Z_o}{\partial t} > 0$. Pettersson points out when this term gets large, it will be opposed by diabatic heating + adiabatic/stratification terms, which also influence temperature.

c) $S \equiv \ln\left(\frac{P_o}{p}\right)\overline{\omega(\Gamma_a - \Gamma)} \Rightarrow$ stratification or adiabatic term

Complicated term, as several factors are included: vertical motion (+/−), static stability, and moisture effects. The net effect of this term requires significant amounts of moisture and vertical motion to exist to be a factor.
⟶ impact during mature stage of cyclone. (Note : different in tropics, whic are conditionally unstable. There this term important during development stage as well)
* ⟶ usually $\frac{\partial Z_o}{\partial t} < 0$.

d) $H \equiv \ln\left(\frac{P_o}{p}\right)\frac{1}{C_p}\overline{\frac{dW}{dt}} \Rightarrow$ diabatic heating term

Heating must be differential for ∇^2 to exist. Good examples of differenti heating include monsoon region, sea breezes, etc. This effect felt for development + decay. One example of mid-lat cyclolysis in winter cyclone which move from over warm water to over cold land (eg E.Pacific → NW US

1/1

4501

MET? : Ruscher (1 hour)

- One form of the Petterssen development equation is given by:

$$\frac{\partial Z_0}{\partial t} = A_Z - \frac{R}{f}\nabla^2\left(\frac{g}{R}A_{\Delta Z} + S + H\right) - \mathbf{V}_0 \cdot \nabla Z_0 - Z(\nabla \cdot \mathbf{V}) \qquad (1)$$

where

$$A_Z \equiv -\mathbf{V} \cdot \nabla Z$$

$$A_{\Delta Z} \equiv -\mathbf{V} \cdot \nabla(\Delta Z)$$

$$S \equiv \ln\left(\frac{p_0}{p}\right)\overline{\omega(\Gamma_a - \Gamma)}$$

$$H \equiv \ln\left(\frac{p_0}{p}\right)\frac{1}{c_p}\overline{\frac{dW}{dt}}$$

and a subscript 0 refers to the surface and non-subscripted terms refer to some upper level in the troposphere.

(1) How is cyclogenesis defined with this equation?

(2) At what level in the troposphere do we evaluate the terms on the right-hand side of Eq. (1)? One term in (1) can immediately be eliminated. Which one is it?

(3) Which other term in (1) is normally ignored for most synoptic scale situations?

(4) Discuss each of the remaining terms of the development equation, and give an illustration of how each term may contribute to cyclogenesis. Include sketches or descriptions of processes which would lead to cyclogenesis.

look at the attached note

Petterssen development eq.

4501

MET? (Mid-latitude Cyclones): Ruscher (40 minutes)

- Petterssen speaks of cyclogenesis in terms of his well-known "development equation"

$$\frac{\partial Z_0}{\partial t} = A_Z - \frac{R}{f}\nabla^2\left(\frac{g}{R}A_{\Delta Z} + S + H\right)$$

where the following definitions are used:

vorticity advection: $A_Z \equiv -\mathbf{V}\cdot\nabla Z$

thickness advection: $A_{\Delta Z} \equiv -\mathbf{V}\cdot\nabla(\Delta Z)$

stratification: $S \equiv \ln\left(\frac{p_0}{p}\right)\overline{\omega(\Gamma_a - \Gamma)}$

heating: $H \equiv \ln\left(\frac{p_0}{p}\right)\frac{1}{c_p}\overline{\frac{dW}{dt}}$

(a) Define each of the terms separately, including symbols used, and including the level at which each term is valid. (b) How would you define cyclogenesis, based on the equation above? (c) Give a thorough physical explanation of each term in the development equation, and give an example of a situation where cyclogenesis is supported. Include diagrams or sketches of appropriate variables.

look at the attached note

note

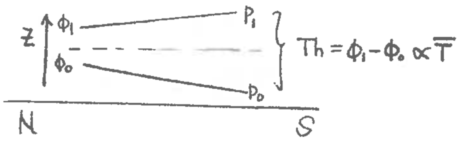
The Sutcliffe-Petterssen Development equation

Includes : formation
changes of intensity } of the system
propagation (motion)

hydrostatic
QG > approximation

Their development is due to ageostrophic flow
(DIV + vertical motion)

Deduce from QG flow, any useful indications about ageostrophic flow



$\zeta \sim \zeta_g = \frac{1}{f}\nabla^2\Phi = \nabla^2\psi \begin{pmatrix}\Phi = gz \\ \psi = \frac{\Phi}{f}\end{pmatrix}$

QG approx

Vorticity eq

$\frac{d\eta}{dt} = -\eta D + \cancel{\text{tilting}}$ where $\eta = \zeta + f$

$\frac{\partial\zeta}{\partial t} = -V\cdot\nabla\eta - \eta D - \cancel{\omega\frac{\partial\zeta}{\partial p}}$

$\shortparallel$

$\left(\frac{\partial}{\partial t}\nabla^2\Phi\right)$

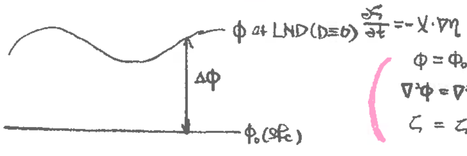
1st law

$$\frac{\partial T}{\partial t} = \underbrace{-V\cdot\nabla T}_{\text{advective rate of change of T with time}} + \underbrace{\omega(\Gamma_d - \Gamma)}_{\text{adiabatic heating/cooling rate}} + \underbrace{H}_{\text{diabatic process including}}$$

$\omega(\Gamma_d - \Gamma) = \left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p}\right) \equiv S$

$H = \frac{1}{q}\frac{dW}{dt}$

diabatic process including (LH, SH, Condensation & evaporation)



$\frac{\partial\zeta}{\partial t} = -V\cdot\nabla\eta$

$\phi = \phi_0 + \Delta\phi$

$\nabla^2\phi = \nabla^2\phi_0 + \nabla^2(\Delta\phi)$

$\zeta = \zeta_0 + \zeta_T$

then $\frac{\partial\zeta_0}{\partial t} = \boxed{\frac{\partial\zeta}{\partial t}} - \frac{\partial\zeta_T}{\partial t}$ ← $-V\cdot\nabla\eta$

$\Delta\phi = R\overline{T}\ln\left(\frac{p_0}{p}\right)$

Taking average on thermodynamic eq

$\frac{\partial\overline{T}}{\partial t} = -\overline{V}\cdot\nabla\overline{T} + \overline{S}\,\overline{\omega} + \overline{H}$

Noting that $\Delta\phi = R\overline{T}\ln\left(\frac{p_0}{p}\right)$

$\frac{\partial}{\partial t}(\Delta\phi) = -\overline{V}\cdot\nabla(\Delta\phi) + \overline{S}\,\overline{\omega}\left(R\ln\left(\frac{p_0}{p}\right)\right) + \overline{H}R\ln\left(\frac{p_0}{p}\right)$

Taking $\frac{1}{f}\nabla^2$ through

$$\frac{\partial\zeta_T}{\partial t} = \frac{1}{f}\nabla^2\Big[-\underline{V}\cdot\nabla\big(\underset{\hookrightarrow g\Delta Z}{\boxed{\Delta\Phi}}\big) + \underbrace{R\ln\left(\frac{p_0}{p}\right)\overline{S}\,\overline{\omega}}_{\equiv S} + \underbrace{R\ln\left(\frac{p_0}{p}\right)\overline{H}}_{\equiv H}\Big]$$

since $\frac{\partial\zeta_0}{\partial t} = \frac{\partial\zeta}{\partial t} - \frac{\partial\zeta_T}{\partial t}$

$$\frac{\partial\zeta_0}{\partial t} = \underbrace{-V\cdot\nabla\eta}_{\text{at LND}} - \frac{R}{f}\nabla^2\left[\frac{g}{R}A_{\Delta Z} + S + H\right]$$

This is the eq for the vorticity tendency at sea-level. In this eq, the first ter on RHS is vorticity advection at LND, 2nd term represents thickness advectio S indicates stratification and H denotes the diabatic heating term.

Sol)

(a)

$\frac{\partial Z_0}{\partial t}$: this is the residual term in the equation It represents the vorticity tendency at the surface (1000mb level), where Z_0 is the vorticity at the surface The forcing terms on the RHS of the eq contribute to cyclonic development if $\frac{\partial Z_0}{\partial t} > 0$

$A_Z = -V\cdot\nabla Z$: This represents the vorticity advection at the level of non-divergence (LND) which is one of the important term for cyclone developme if there is positive vorticity advection

$A_{\Delta Z} = -V\cdot\nabla(\Delta Z)$: ΔZ is the thickness between LND and the 1000mb. This term represents the advection of thickness below LND

$S \equiv \ln(\frac{P_0}{P})\overline{\omega(\Gamma_a-\Gamma)}$: this is the stratification term where $\overline{\omega}$ is the vertical motion ($\omega = \frac{dp}{dt}$, $\omega = -\rho g W$). $\omega<0$: upward motion, $\omega>0$: downward motion. $\Gamma_a - \Gamma$ is the stability term, where Γ_a is dry lapse rate of the atmosphere and Γ is the sounding $\frac{dT}{dz}$. $\ln(\frac{P_0}{P})$ is logarithm of 1000mb/pressure of LND. This term represents the stratification below LND

$H \equiv \ln(\frac{P_0}{P})\frac{1}{C_p}\frac{d\overline{W}}{dt}$: $\frac{1}{C_p}\frac{d\overline{W}}{dt}$ represents the diabatic heating which could result from cooling or warming due to the surface differential heating and the heat released in the atmosphere

(b)

Cyclogenesis is defined as any development or strengthening of cyclonic circulation in the atmosphere where it's associated with upper-level vorticity advection and vertical distribution of convergence and divergence and change in surface pressure in the surface ~~pressure tendency~~ ⟶ deepening

(c) Physical explanation of each term.

① Heating term : $-\frac{R}{f}\nabla^2 H$

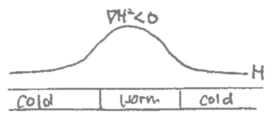
If heat is introduced to the lower troposphere

$H>0 \longrightarrow \nabla^2 H<0 \longrightarrow -\frac{R}{f}\nabla^2 H>0 \longrightarrow \frac{\partial Z_0}{\partial t}>0$ Cyclonic (development)

Conversely

$H<0 \longrightarrow \frac{\partial Z_0}{\partial t}<0$ ⟶ anticyclonic development

So it is clear that heat contributes to the cyclonic development. This term supports the development as follows



* Monsoon circulation, sea breeze, Katabatic wind where the heat generate circulation.
* the anticyclone during winter move over the east coast of the U.S. So the heating of the air over the warm water change the negative vorticity to positive vorticity. So the anticyclone weakens or deforms due to the cyclonic vorticity.
* The low intensify when it moves over the Great Lakes (warm water) in winter
* This term supported cyclone to be explosive when it move over the Gulf water stream offshore of the east coast of the U.S

② Stratification term : $-\frac{R}{f}\nabla^2 S = -\frac{R}{f}\nabla^2[\ln(\frac{P_0}{P})\overline{\omega(\Gamma_a-\Gamma)}]$

$\ln(\frac{P_0}{P})>0$ since $P_0>P$

$\overline{\omega}<0$ upward motion

>0 downward motion

$\Gamma_a - \Gamma>0$ in the midlatitudes

This term tend to work against the development because $\Gamma_a-\Gamma>0$ unless there is a saturated air $\Gamma_s-\Gamma<0$

$$\left.\begin{array}{l}\Gamma_a-\Gamma>0\\ \overline{\omega}<0\\ \ln(\frac{P_0}{P})>0\end{array}\right)\ S<0 \longrightarrow \nabla^2 S>0 \longrightarrow -\frac{R}{f}\nabla^2 S<0 \longrightarrow \frac{\partial Z_0}{\partial t}<0$$

anticyclonic development

So, this term does not support cyclonic development

③ Thickness advection term : $-\frac{R}{f}\nabla^2(\frac{g}{R}A_{\Delta z})$

$-\underline{V}\cdot\nabla(\Delta z)$ will be maximum with warm advection

so $\nabla^2 A_{\Delta z}$ = minimum $\longrightarrow -\frac{g}{f}\nabla^2 A_{\Delta z}$ = maximum and hence $\frac{\partial Z_0}{\partial t}>0$

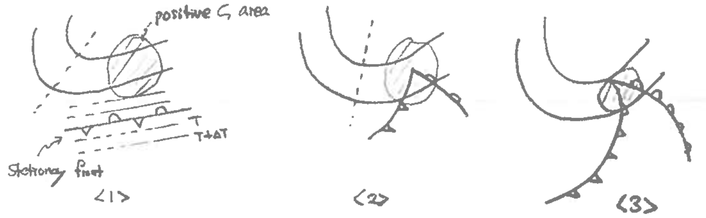
with warm advection ahead of cyclone in the warm frontal zone there is an increase of cyclonic vorticity. At rear of cyclone with cold advection there is a maximum of negative vorticity. So, the cyclone move to the rising cyclonic vorticity (to the region of warm advection

* This term is not important at the initial development but it is important later
* The adiabatic and diabatic (term S & H) work against this term because in the secondary circulation sense, the rising/sinking partly cancel the temperature advection

④ Vorticity advection term : $-\underline{V}\cdot\nabla Z$ at LND

The vorticity adv term is very important for cyclonic development at the initial development (there is no thickness advection, no effect of stratification or diabatic

- So the only term which contribute to the initial development is vorticity adv. at upper level (say, 500 mb)



* Petterssen's development eq.

• Development

A simplified form of the vorticity eq is obtained after typical large scale analysis is applied to the full eq.

$$\frac{d(\zeta+f)}{dt} = -\underbrace{(\zeta+f)}_{\zeta_a}(\nabla\cdot\underline{V})$$

From the above eq we see that in the absence of divergence absolute vorticity is conserved. Convergence leads to vorticity production. Divergence leads to vorticity destruction. We may define vorticity as

$$\zeta = \frac{\text{Circulation}(C)}{\text{enclosed area}(A)}$$

By the Circulation theorem of Bjerknes

$$C = \oint \underbrace{V_s}_{\text{component of motion along a streamline}} ds$$

If circulation increases the rate at which flow is moving along the contour increases. An increase in circulation also implies an increase in vorticity. A cyclonic system is said to be developing if $\frac{d\zeta_a}{dt} > 0$. Development or cyclogenesis occurs along with production of vorticity (circulation) and an increase of wind speed in the direction of the motion.

• Dynamics of development

Since development is related to the time rate change of vorticity and divergence, it is useful to obtain expressions for these quantities. We can apply the vorticity eq at the level of non-divergence (LND) and sea level with the barotinicity of the intervening layer being defined by the thermal wind relationships. Petterssen did this. Below we roughly outline how he obtained his development eq for surface cyclogenesis.

Petterssen began with

$$\frac{\partial\zeta}{\partial t} = -\underline{V}\cdot\nabla_p(\zeta+f) = \frac{\partial\zeta_o}{\partial t} + \frac{\partial\zeta_T}{\partial t}$$

Where unsubscripted variables refer to some level in the upper troposphere (LND) and "0" refers to the sfc. ζ_T is the thermal vorticity. It is the vorticity of the thermal wind and may be written as

$$\zeta_T = \frac{g}{f}\nabla^2(\underbrace{\Delta Z}_{\text{thickness of layer between sfc and specified upper level layer.}})$$

The vorticity at the upper level may be written as

$$\zeta = \zeta_o + \zeta_T$$

Since $\underline{V} = \underline{V}_o + \underline{V}_T \longleftarrow$ thermal wind relation

The next step is to use the relationship

$$\frac{\partial\zeta_T}{\partial t} = \frac{g}{f_o}\nabla_p^2\frac{\partial(\Delta Z)}{\partial t}$$

and substitute for $\frac{\partial}{\partial t}(\Delta Z)$ the vertically integrated average thickness eq shown below

$$\frac{\partial}{\partial t}(\Delta Z) = \underset{\substack{\uparrow \\ \text{horizontal} \\ \text{advection}}}{-\underline{V}_g\cdot\nabla_p(\Delta Z)} + \underset{\substack{\uparrow \\ \text{adiabatic} \\ \text{cooling/heating}}}{\left(\frac{R_d\overline{T}\Delta p}{g\overline{p}}\right)\overline{\omega}} + \underset{\substack{\uparrow \\ \text{diabatic} \\ \text{effects.}}}{\left(\frac{R_d\overline{T}\Delta p}{gC_p\overline{p}}\right)\overline{\dot{Q}}}$$

So

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\zeta_o}{\partial t} + \underbrace{\frac{g}{f_o}\nabla_p^2\left[\overline{A_T}+\overline{S}+\overline{H}\right]}_{\frac{\partial\zeta_T}{\partial t}} = -\underline{V}\cdot\nabla_p(\zeta+f)$$

Solve for $\frac{\partial\zeta_o}{\partial t}$

$$\frac{\partial\zeta_o}{\partial t} = -\underline{V}\cdot\nabla_p(\zeta+f) - \frac{g}{f_o}\nabla_p^2\left[\overline{A_T}+\overline{S}+\overline{H}\right] - (\zeta+f)(\nabla\cdot\underline{V})$$

(we would retain this term if the upper level were not the LND)

Recall that development is defined in terms of the sign of $\frac{d\zeta_o}{dt}$

Since $\frac{d\zeta_o}{dt} = \frac{\partial\zeta_o}{\partial t} + \underline{V}_o\cdot\nabla(\zeta_o+f)$

$$\frac{\partial\zeta_o}{\partial t} = -\underline{V}\cdot\nabla_p(\zeta+f) - \frac{g}{f_o}\nabla_p^2\left[\overline{A_T}+\overline{S}+\overline{H}\right] \underset{\substack{\uparrow \\ \text{apply at LND} \\ \text{so } \nabla\cdot\underline{V}=0}}{-(\zeta+f)(\nabla\cdot\underline{V})} \underset{\substack{\uparrow \\ \text{assume advection} \\ \text{of vorticity at} \\ \text{sfc is negligable}}}{-\underline{V}_o\cdot\nabla(\zeta_o+f)}$$

• Petterssen's development eq

$$\frac{\partial\zeta_o}{\partial t} = -\underline{V}\cdot\nabla_p(\zeta+f) - \frac{g}{f_o}\nabla_p^2\left[\overline{A_T}+\overline{S}+\overline{H}\right]$$

where

$\overline{A_T} \equiv -\underline{V}_o\cdot\nabla_p(\Delta Z) \longleftarrow$ low level thickness (temperature) advection

$\overline{S} \equiv \frac{R_d\overline{T}\Delta p}{g\overline{p}}\overline{\omega} \longleftarrow$ adiabatic effects

$\overline{H} \equiv \frac{R_d\overline{T}\Delta p}{gC_p\overline{p}}\overline{\dot{Q}} \longleftarrow$ diabatic effects

The development process is fairly complex. It depends on vorticity adv. + 3 thermal terms.

① $-\underline{V}\cdot\nabla_p(\zeta+f) \equiv$ vorticity advection at LND

② $-\frac{g}{f}\nabla^2\overline{A_T} \propto +\frac{g}{f}\overline{A_T} \equiv$ localized thickness advection by surface wind

③ $-\frac{g}{f}\nabla^2\overline{S} \propto +\frac{g}{f}\overline{S} \equiv$ localized adiabatic cooling/heating

④ $-\frac{g}{f}\nabla^2\overline{H} \propto +\frac{g}{f}\overline{H} \equiv$ " diabatic heating/cooling

We note that $\frac{\partial\zeta_o}{\partial t} > 0$ (surface development) occurs when

① PVA at upper level (here, the LND)

② low level warm air advection in localized regions

③ adiabatic term ⟵ complicated!

④ localized regions of heating.

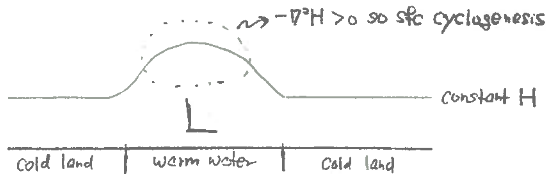
Now lets consider the thermal terms more carefully

Diabatic effects. $\overline{H} = \frac{R_d}{gC_p}\frac{\Delta p}{\overline{p}}\overline{T}\,\overline{\dot{Q}}$ and $-\frac{g}{f}\nabla^2\overline{H} \Rightarrow +\frac{g}{f}\overline{H} \begin{matrix} >0 & \zeta_o\uparrow \\ <0 & \zeta_o\downarrow \end{matrix}$

Heat sources generate cyclonic vorticity. Cold surfaces are regions of anticyclogenesis. We stress, however, that this heating/cooling must be nonuniform (for uniform fields the Laplacian is zero). Examples where diabeti effects can be large include monsoon regions and areas where land-sea breez or mountain-vally breezes are common. This effect can also be important in air mass modification (eg, the transformation of cP into mT) as cold air moves warm water resulting in lake effect snows downwind in Great Lakes, coastal cyclogenesis over the western Atlantic + Pacific oceans near the Gulf Strea + Kuroshio current, respectively.

Anticyclogenesis can be caused for the opposite case. Examples of cyclolysis include wintertime cyclolysis over the Pacific NW and Europe as storms leav relatively warm waters and move over colder land. This effect also works for anticyclones as we often observe anticyclones stall as they move into the easter US in winter, due to the fact that at their leading edge a heat source is being encountered, thus opposing anticyclonic development.

The diagram below illustrates these effects



Adiabatic effects $\overline{S} = \frac{R_d}{g}\frac{\Delta p}{\overline{p}}\overline{\sigma}\,\overline{\omega}$, $-\frac{g}{f}\nabla^2\overline{S} \Rightarrow +\frac{g}{f}\overline{S} \begin{matrix} >0 & \zeta_0\uparrow \\ <0 & \zeta_0\downarrow \end{matrix}$

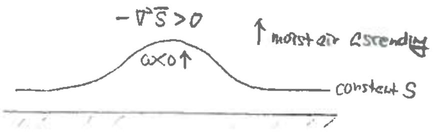
We often refer to this as the stratification term. For development to occur due to the stratification term, we must have nonzero vertical motion ($\overline{\omega} \neq 0$). Generally, we associate upward movement with cyclonic development ($\frac{d}{dt}(\zeta_0+f)>0$) but this will not in general be the case. The atmosphere is normally statically stable ($\overline{\sigma}>0$) and so the product $\overline{\sigma}\,\overline{\omega}<0$ for rising motion. Negative $\overline{\sigma}\,\overline{\omega}$ implies $\overline{S}<0$ and so $\frac{d}{dt}(\zeta_0+f)<0$. At first this seems incorrect, but recall that upward motion leads to adiabatic cooling ⇒ a damping effect.

This all changes if the atmosphere is saturated. Then the dry static stability is negative. For rising motion in a saturated environment $\overline{\sigma}\,\overline{\omega}>0$ so $(\zeta_0+f)\uparrow$ with time. The latent heat release that accompanies condensation in a rising, saturated parcel leads $\overline{S}$ to be of the same sign as $\overline{H}$. This will not happen until significant cloudiness has already been developed within the cyclone. Also, upward motion is usually not large in the incipient stages of cyclonic development. Therefore, the stratification term will not in general contribute to the initial development of a cyclone, but will be associated with its further strengthening later in time. (This remarks applies to mid-lat. cyclones but in the conditionally unstable tropics this term is necessary for the development of tropical cyclones)

The main effect of condensation is to minimize the negative effect of stability. Provided other terms in the development eq are favorable, cyclonic development will occur when there is

① a minimum instability
② a maximum of moisture
③ a low condensation level to get clouds started.

The below diagram illustrates points made above



Thickness advection $\overline{A_T} = -V_0\cdot\nabla_p(\Delta z)$, $-\frac{g}{f}\nabla^2\overline{A_T} \Rightarrow +\frac{g}{f}\overline{A_T} \begin{matrix} >0 & \zeta_0\uparrow \\ <0 & \zeta_0\downarrow \end{matrix}$

In practice, this term is not important for initial development but is very important for maintenance of the cyclonic storm. Max thickness advection (max warm air adv.) will foster the development of cyclonic systems. According to Petterssen, when this term becomes significantly large, this term will be opposed by the diabatic and adiabatic terms which also influence temperature.

Vorticity advection PVA → $\zeta_0\uparrow$: NVA → $\zeta_0\downarrow$ (vorticity adv at LND)

PVA at the LND leads to development at sea level. According to Petterssen this term (upper level vorticity advection) is most important for providing for the initial development. To see why this is so, examine vorticity advection in natural coordinates

$$\text{absolute vorticity} \equiv \zeta + f = VK_s - \frac{\partial V}{\partial n} + f$$

VK_s → curvature term; $\frac{\partial V}{\partial n}$ → shear term

where K_s is curvature of streamline of flow $K_s = \frac{1}{R_s}$

The advection of $\zeta+f$ is

$$-V\frac{\partial}{\partial s}(\zeta+f)$$ ← advection of absolute vorticity along the streamline.

But we have an expression for $\zeta+f$. Thus

$$\frac{\partial}{\partial s}(\zeta+f) = \frac{\partial}{\partial s}\left(VK_s - \frac{\partial V}{\partial n} + f\right)$$
$$= K_s\frac{\partial V}{\partial s} + V\frac{\partial K_s}{\partial s} - \cancel{\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial n}\right)} + \cancel{\frac{\partial f}{\partial s}}$$

~0 (Cross flow shear varies little along the streamline even though it is usually quite large in the region of interest)

~0 ↳ Small if streamline is nearly along latitude circle.

Thus advection of absolute vorticity in natural coord is approx.

$$-V\frac{\partial V}{\partial s} \simeq -V\left(K_s\frac{\partial V}{\partial s} + V\frac{\partial K_s}{\partial s}\right)$$
$$\sim -VK_s\frac{\partial V}{\partial s} - V^2\frac{\partial K_s}{\partial s}$$
$$\sim -V^2\left(K_sK_n + \frac{\partial K_s}{\partial s}\right)$$

where $K_n = \frac{1}{V}\frac{\partial V}{\partial s}$ is the normalized curvature normal to the streamline.

The thing to notice is that vorticity advection is proportional to the wind speed squared. Providing the configuration of the contour is favorable, this term is crucial for initial development.

Digression : Self development

Whether or not self development may occur, we observe that cyclones experience a life cycle which includes their ultimate demise. The stratification term plays an important role in this regard

Look at the stratification (S) term. It involves the average correlation between $\overline{\omega}$ and $\overline{\sigma}$. As we saw, normally in cyclonic systems $\overline{\omega}<0$, $\overline{\sigma}>0$ so $S<0$ and $\frac{d}{dt}(\zeta_0+f)<0$ which opposes development. We visualize how this works by remembering that upward motion is associated with adiabatic cooling. Also, the magnitude of the divergence is proportional to the storm intensity, we know that vertical motion is related to divergence ($\frac{\partial\omega}{\partial p} = -\nabla\cdot V$). So with amplification we get more upward motion which makes S more strongly negative and hence inhibits generation of cyclonic vorticity at the sfc. (Since $\frac{d}{dt}(\zeta_0+f)<0$).

We see that in the very process of amplification their exist mechanisms (the stratification term) which limit development. That is, cyclonic development is a self limiting process.
↓
As storm strengthens the surface pressure drops

$$\frac{\partial P_s}{\partial t} = -\int_0^{P_s}\nabla\cdot V\,dp$$

This implies a net divergence of mass out of the column over the storm interior. We have strong low-level convergence ⇒ generating cyclonic vorticity at the sfc and strong upper level divergence, Via the continuity eq $\nabla\cdot V = -\frac{\partial\omega}{\partial p}$, we see that as the divergence increases in magnitude so must the magnitude of vertical velocity (rising motion here). Rising motion is associated with adiabatic cooling which opposes further development. We noted that latent heat release countered this cooling in saturated ascent. As the system matures the moisture supply can be exhausted or cut off and so adiabatic cooling dominates. This tends to occur in the later stages of the cyclone life cycle. We see the self limiting character of cyclones

Development of Cyclones and Anticyclones

Petterssen/Sutcliffe Formulation

References

Palmén, E. and C. W. Newton, 1969: *Atmospheric Circulation Systems*. New York, Academic Press, Chapter 11.

Petterssen, S., 1956: *Weather Analysis and Forecasting I*, New York, McGraw-Hill, Chapter 16.

Sutcliffe, R. C., 1947: On the dynamics of development. *Quart. J. Roy. Meteorol. Soc.*, **73**, 370-383.

Semantics

cyclogenesis[1]***:*** Any development or strengthening of cyclonic circulation in the atmosphere; the opposite of cyclolysis. It is applied to the development of cyclonic circulation where previously it did not exist as well as the intensification of existing cyclonic flow.

The Meaning of Development

A very simplified form of the vorticity equation is obtained after a typical large-scale scale analysis is applied to the full equation, namely

$$\frac{1}{\zeta_a}\frac{d\zeta_a}{dt} = -\nabla \cdot \mathbf{V} \qquad (1) \qquad \frac{1}{\zeta_a}\frac{d\zeta_a}{dt} = -\nabla \cdot \underline{V}$$

which says that in the absence of divergence, absolute vorticity is conserved. Also, convergence leads to vorticity production. Vorticity is defined as

$$\zeta \equiv \frac{\text{circulation (C)}}{\text{area (A)}} \qquad (2)$$

By the Circulation Theorem of Bjerknes,

$$C \equiv \oint V_s \, ds \qquad (3)$$

if the component of motion increases along a streamline, circulation increases, which leads to an increase in the vorticity. A (cyclonic) system is said to be developing if d[[zeta]]$_a$/dt > 0. Development, or cyclogenesis, occurs along with production of vorticity (circulation) and an increase of wind speed in the direction of the motion.

The Dynamics of Development

Because development is related to the time rate of change of air circulating around a cyclone or anticyclone, we can express the concept of development in terms of quantities like surface pressure, vorticity, and divergence, it is useful to obtain expressions for these quantities. We will follow Petterssen in this treatment and develop a prognositic equation for the surface vorticity tendency. We can apply the vorticity equation at sea-level and at the level of non-divergence (LND), with the baroclinicity of the intervening layer being defined by the thermal wind relationship.

Without going into all the gory details, the Petterssen development equation may be written

$$\frac{dZ_0}{dt} = -D_0 Z_0 = -\frac{\partial \zeta_T}{\partial t} - \mathbf{V} \cdot \nabla \zeta_T - \mathbf{V}_T \cdot \nabla Z_0 \qquad (4) \quad \frac{dZ_0}{dt} = -D_0 Z_0 = -\frac{\partial \zeta_T}{\partial t} - \mathbf{V} \cdot \nabla \zeta_T - \mathbf{V}_T \cdot \nabla Z_0$$

where [[Zeta]] = [[zeta]]$_a$, D symbolizes divergence, subscript T refers to the thermal wind (on V) or thermal vorticity (on [[zeta]]), subscript 0 refers to sea level (actually 1000 mb), and no subscript refers to the LND.

Thermal vorticity is interpreted as the vorticity of the thermal wind, and may be written as

$$\zeta_T = \frac{g}{f} \nabla^2 (\Delta z) \qquad (5) \quad \zeta_T = \frac{g}{f} \nabla^2 (\Delta z)$$

where [[Delta]]z is the thickness of the layer bounded by the LND and the surface. The vorticity at the LND ([[zeta]]) is simply the sum of the vorticity at the surface ([[zeta]]$_0$) and the thermal vorticity ([[zeta]]$_T$),

$$\zeta = \zeta_0 + \zeta_T \qquad (6)$$

because $\mathbf{V}_T = \mathbf{V} - \mathbf{V}_0$.

By using the thermal wind relationship in (4), we can ultimately write the equation for vorticity tendency at sea-level as

$$\boxed{\frac{\partial Z_0}{\partial t} = A_Z - \frac{R}{f} \nabla^2 \left(\frac{g}{R} A_{\Delta z} + S + H \right)} \qquad (7) \quad \frac{\partial Z_0}{\partial t} = A_Z - \frac{R}{f} \nabla^2 \left(\frac{g}{R} A_{\Delta z} + S + H \right)$$

where the following definitions are used:

vorticity advection: $A_Z \equiv -\mathbf{V} \cdot \nabla Z$

thickness advection: $A_{\Delta z} \equiv -\mathbf{V} \cdot \nabla (\Delta z)$

stratification: $S \equiv \ln\left(\frac{p_0}{p}\right) \overline{\omega (\Gamma_a - \Gamma)}$

heating $H \equiv \ln\left(\frac{p_0}{p}\right) \frac{1}{C_p} \overline{\frac{dW}{dt}}$

Vorticity production is the sum of the vorticity tendency (7) and the vorticity advection at sea-level (-$\mathbf{V}_0$ * [[gradient]][[Zeta]]$_0$) which is very small and usually neglected.
$-\mathbf{V}_0 \cdot \nabla Z_0$

The development process is thus fairly complicated, with one term involving vorticity advection at the LND and the other three terms (called thermal terms) interrelated. Development at sea-level occurs when there is an imbalance between vorticity advection at the LND and the Laplacian of the thermal terms. Cyclogenesis or development occurs when A$_{[[Zeta]]}$ is positive and the [[gradient]]^2 ( ) terms are negative (note the minus sign prefixing the term).

The Laplacian (a review!)

Recall that the geostrophic vorticity (which is approximately the relative vorticity) may be written as

$$\zeta_g = \frac{g}{f} \nabla_p^2 z \approx \zeta$$

The relative vorticity is positive in troughs ([[zeta]] > 0 when z is a minimum) and negative in ridges ([[zeta]] < 0 when z is a maximum). A similar interpretation may be given to the Laplacian terms in (7), i.e. when the function A is a maximum, [[gradient]]^{2}A is positive. The Laplacian is a second derivative and so represents curvature. Therefore we must have some large change in magnitude of the function A for there to be a large Laplacian term.

The finite difference form of the Laplacian applied to the grid below is

$$\nabla^2 A = \frac{A_1 + A_2 + A_3 + A_4 - 4A_0}{d^2}$$

where d is the spacing between grid points.

* A_2

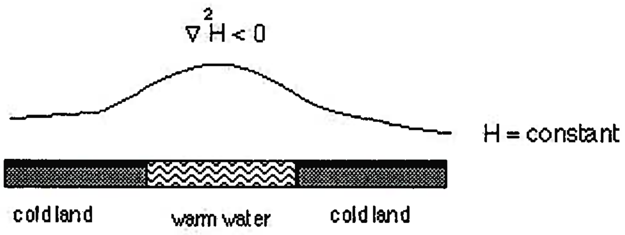
* A_1 * A_0 * A_3

* A_4

Diabatic effects

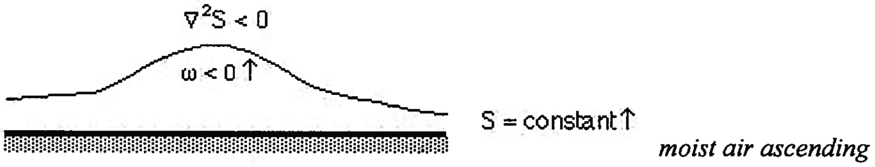
Heat sources generate cyclonic vorticity (H > 0 => [[gradient]]^{2}H < 0 => -[[gradient]]^{2}H > 0 => d[[Zeta]]$_0$/dt > 0) and cold surfaces are regions of anticyclogenesis. However, the heating (cooling) effect must *not* be uniform, otherwise the Laplacian is zero! Examples of areas where these influences are large include monsoon regions and areas where land-sea breezes or mountain-valley breezes are common. This effect can also be important in air mass modification (e.g. the transformation of cP into mT) as cold air moves over warm water (resulting in lake effect snows downwind of the Great Lakes, and coastal cyclogenesis over the western Atlantic and Pacific Oceans near the Gulf Stream and Kuroshio Current, respectively).

Anticyclogenesis can be caused for the opposite case. Examples of cyclolysis include wintertime cyclolysis over the Pacific Northwest and Europe as storms leave the (*relatively*) warm waters and move over colder land. This effect also works for anticyclones, as we often observe anticyclones to stall as they move into the eastern US in winter, due to the fact that at their leading edge, a heat source is being encountered, thus opposing anticyclonic development.



Effects of stratification

Remember [[omega]] and w have opposite signs. For development to occur due to the stratification term, we must have non-zero vertical motion ([[omega]] != 0). We normally associate upward motion with cyclonic development (d[[Zeta]]$_0$/dt > 0), *but* this will not in general be the case! The atmosphere is normally statically stable ([[Gamma]]$_a$ > [[Gamma]]), and the product of [[omega]] and ([[Gamma]]$_a$ - [[Gamma]]) is negative, which leads to a *decrease* of d[[Zeta]]$_0$/dt > 0 with time. This should make sense if you recall that upward motion leads to adiabatic *cooling*. If latent heat is being released upon ascent, however, we use [[Gamma]]$_s$ rather than [[Gamma]]$_a$, both factors are negative, and this term (S) is of the same sign as H and contributes to development. This will not happen until significant cloudiness has already been developed within the cyclone. Also, upward motion is usually not large in the incipient stages of cyclonic development. Therefore, it will not in general contribute to the *initial* development of a cyclone, but will be associated with its further strengthening later in time. [All of this discussion works well for mid-latitude cyclones, but not for tropical cyclones, which *require* this term for development!]



The main effect of condensation is to minimize the negative effect of stability. Provided other effects (terms in Eq. 7) are favorable, cyclonic development will occur when there is a minimum in stability, a maximum of moisture, and a low condensation level to help the clouds get started.

The effects of thickness advection

In practice, this term is very important for maintenance of the cyclonic storm, but not for initial development. Maximum thickness advection {maximum warm advection, max -**V** * [[gradient]]T > 0} will foster development of cyclonic systems. According to Petterssen, when this term becomes significantly large, it will be opposed by the diabatic and adiabatic terms (H and S, respectively), which also influence temperature.

Digression #2: The self-development process and the limitations of the stratification term (after Sutcliffe)

Whether or not self-development may occur, we observe that cyclones experience a life cycle which includes their ultimate demise. The stratification term plays an important role in this regard. How does the stratification term work to oppose the effects of development of a cyclonic storm? Or, why do cyclones die? Let's see ...

Look at the S term again; it involves the average correlation between [[omega]] and [[Gamma]]$_a$-[[Gamma]]. Normally in cyclonic systems [[omega]] < 0 and [[Gamma]]$_a$-[[Gamma]] > 0 so that S < 0 and d[[Zeta]]$_0$/dt < 0 (which *opposes* development). We can visualize how this works be remembering that upward motion is associated with adiabatic cooling. Also, the magnitude of the divergence is proportional to the amplitude of the storm (from P&N Ch. 6), and we know that vertical motion is related to the divergence so with amplification, we get more upward motion, which makes S more strongly negative (minimum) and hence inhibits generation of cyclonic vorticity at the surface. In fact during the life cycle of the cyclone, the stratification term becomes more important such that cyclonic scale development is a self-limiting process.

The effects of vorticity advection

PVA at the LND leads to development at sea level, according to the development equation. This is the big one and is important for providing the initial development, according to Petterssen. The role this plays can be further explored by looking at an alternate expression for vorticity advection using natural coordinates:

$$Z = VK_s - \frac{\partial V}{\partial n} + f \qquad (8)$$

where V is the wind speed, K_s is the curvature along a streamline and $^{\text{[[partialdiff]]V}}/_{\text{[[partialdiff]]n}}$ is the shear term. Usually the shear varies little along the streamline even though it is usually quite large in the region of interest. We can then write the advection of [[Zeta]] as

$$A_z = \left(V \frac{\partial K_s}{\partial s} + K_s \frac{\partial V}{\partial s} \right) = - V^2 \left(\frac{\partial K_s}{\partial s} + K_s K_n \right) \qquad (9)$$

Note the important result: vorticity advection is proportional to the wind speed *squared*! I wonder if this has applications for jet streams? Providing the configuration of the contours (streamlines) is favorable, this is *crucial* for initial development, according to the theory.

To conclude we present two quotes from Petterssen (Ch. 16):

* *"... the appreciable amounts of vorticity advection aloft are limited to small areas, separated by vast areas of insignificant values of vorticity advection."*

* *"Cyclone development at sea level occurs when and where an area of appreciable vorticity advection in the upper troposphere becomes superimposed upon a slowly moving or quasi-stationary front at sea level."* (Ah, if were only that simple!)

barotrophic / baroclinic instability and atmosphere.

MET? : Ruscher (?, 1995)

- Explain and contrast barotropic and baroclinic instability as they pertain to synoptic scale disturbances. Be sure to include information about both the theoretical and practical aspects of these instability mechanisms. Also, describe the difference between a barotropic and baroclinic atmosphere.

look at other study notes!

1/

(general)
MET? : Ruscher (60 minutes) *

- Tallahassee has a reputation for being colder than the surrounding areas on certain nights. In fact, one can often what television weathercasts and hear the weathercaster state that "the temperature will be colder in the low spots like the airport." Do you agree with the above statement (quoted above)? Using your familiarity with the local area, and your knowledge of dynamic, physical, and synoptic meteorology, discuss the basis for your agreement or disagreement in terms that an introductory meteorology (Met 1010) student could understand. Include some sketches or diagrams that would be. **helpful.**

Sol)

The temperature at the TLH airport are often colder by several degrees than those at surrounding locations when the area is under clear skies and light winds.

↳ these synoptic conditions maximize the difference between TLH and surrounding area temperatures. The primary cooling mechanism under these conditions is radiational cooling. But why are TLH temp cooler?

① land around airport has been stripped of trees + shrubs. The trees that surround the airport are largely pines. Pines, with their fewer banches, needles, and smaller canopy are less effective in intercepting LW from the sfc. The absence of dense vegetation at the sfc also allow LW to more readily escape. → fewer vegetation

② Soil around airport is a sandy loam. Thus it is very pourous to water. Water has a high heat capacity and is slower to change temperature. Moist soil heats + cools slower than dry soil. The sandy soil in + around the airport is more likely to be dry than very moist. → moisture (sandy loam)

③ There are few manmade structures in and around the airport. Furthermore the airport is far enough southwest of the urban heat island towards downtown TLH that it is not warmed by advection.

→ Paved roads + concrete structures retain daytime heating and thereby slow cooling rates at night. The numerous roads and buildings in developed urban centers create local warm pools ⇒ urban heat islands. → manmaid structures

④ Cool air drainage. The airport is in a lower — area relative to its surrounding. Thus the gravitational, density current type drainage of cold air into the airport area reinforces the cool temperatures. As TLH is not mountainous ⇒ this effect may not be the dominant reason for cold temps at the airport but it is a factor. → Cool air drainage

//

(general, PBL)

MET? : Ruscher (60 minutes)

- A very “hot” issue in Tallahassee right now involves the construction of small-scale medical waste incinerators. These incinerators burn waste around the clock at a temperature of approximately 1500°F. Some in state and local government have concerns about the prospects for local and regional impacts on air quality. Suppose that the county’s emergency planning office wants to evaluate the impacts on air quality at the present incinerator cite, off Springhill Road, and a possible additional site, near Lake Jackson. Discuss in detail the procedures you would recommend to accomplish this, considering likely sources of data. How much confidence do you feel should be ascribed to the results of the evaluation you describe and why?

⟹ Hmmm ....

PBL (6480)

MET? : Ruscher (60 minutes)

- Suppose you were given the code for a three-dimensional numerical weather prediction model of the atmosphere (e.g., the FSU global spectral model). Your task is to adapt the model for use in atmospheric chemical studies. (a) What chemical species (or families) would you consider to be most important to include in short- and medium-range forecast (out to 10 days) runs. Discuss very briefly the significance of each species/family. (b) How would you like to parameterize boundary layer processes involving these chemical species, given that the current model uses standard K-theory? If using the existing K-theory, what assumptions would need to be made? If you would use a different method, describe it. (c) Describe in a paragraph or two a scientific problem which might be suitable for investigation with such a model. What kind of hypothesis would you be testing and what kinds of experiments might you devise?

Hmmm ooo.

1/1

K.E. eq + Ri #

6480 (PBL)

MET? (Turbulence question): Ruscher (1 hour) ⟶ Try to simplify the answer!

✓

- Derive the kinetic energy equation starting with the equation of motion (written as a local tendency of the zonal wind). Show how the Richardson number can be derived from terms of this equation, and give an interpretation for laminar, turbulent, and transitional flows in terms of the Richardson number.

look at my study note !

Sol) Following Einstein's summation notation, the eq. of motion can be written as follows.

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g + f \varepsilon_{ij3} U_j - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}$$

Using hydrostatic, incompressibility + Boussinesq approx, The above eq. can be simplified as

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3}\left[g - \left(\frac{\theta_v'}{\overline{\theta_v}}\right) g\right] + f \varepsilon_{ij3} U_j - \frac{1}{\overline{\rho}}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

Here, we have to derive the eq. for mean winds in a turbulent flow. $U_i = \overline{U_i} + u_i'$, $\overline{P} + p'$ etc.

Sub. the above and after applying the Reynolds averaging, also utilizing the turbulent continuity eq., we have

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3} g + f \varepsilon_{ij3} \overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{U_i}}{\partial x_j^2} \bigcirc\!\!\!\!\!\!\left(-\frac{\partial \overline{u_i' u_j'}}{\partial x_j}\right)$$

Next, the momentum eq for the turbulent departure must be obtained by subtracting the mean part eq. from the original,

$$\frac{\partial u_i'}{\partial t} + \overline{U_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{U_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} = +\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right) g + f_0 \varepsilon_{ij3} u_j'$$
$$- \left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial (\overline{u_i' u_j'})}{\partial x_j} \quad \text{———} ⓐ$$

By $2u_i' *$ ⓐ, Reynolds averaging and scale analysis, the velocity variance eq can be obtained.

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U_j}\frac{\partial \overline{u_i'^2}}{\partial x_j} = +2\delta_{i3}\frac{g\,\overline{u_i'\theta_v'}}{\overline{\theta_v}} - 2\overline{u_i' u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \frac{\partial \overline{u_j' u_i'^2}}{\partial x_j}$$
$$- \frac{2}{\overline{\rho}}\frac{\partial (\overline{u_i' p'})}{\partial x_i} - 2\varepsilon.$$

Here, Let's define TKE

$$\frac{TKE}{m} = \overline{e} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

So

$$\boxed{\frac{\partial \overline{e}}{\partial t} + \overline{U_j}\frac{\partial \overline{e}}{\partial x_j} = \underbrace{\delta_{i3}\frac{g}{\overline{\theta_v}}(\overline{u_i'\theta_v'})}_{Ⅰ} \underbrace{- \overline{u_i' u_j'}\frac{\partial \overline{U_i}}{\partial x_j}}_{Ⅱ} - \frac{\partial \overline{u_j' e}}{\partial x_j} - \frac{1}{\overline{\rho}}\frac{\partial (\overline{u_i' p'})}{\partial x_i} - \varepsilon}$$

↳ TKE budget eq

The ratio of term Ⅰ to term Ⅱ in TKE eq defines flux Ri #.

$$R_f = \frac{\text{buoyancy term}}{-\text{Shear term}}$$

$$\boxed{R_f = \frac{\left(\frac{g}{\overline{\theta_v}}\right)\overline{w'\theta_v'}}{\overline{u_i' u_j'}\frac{\partial \overline{U_i}}{\partial x_j}}}$$

Assume horizontal homogeneity and neglect subsidence,

$$R_f = \frac{\left(\frac{g}{\overline{\theta_v}}\right)\overline{w'\theta_v'}}{\overline{u'w'}\frac{\partial \overline{U}}{\partial z} + \overline{v'w'}\frac{\partial \overline{V}}{\partial z}}$$

For statically unstable flows, Rf is usually negative (remember that the denominator is usually negative). For neutral flows, it is zero. For statically stable flows, Rf is positive.

Rf = +1 is a critical value.

(Flow is turbulent (dynamically unstable) when Rf < +1
(Flow becomes laminar (dynamically stable) when Rf > +1.

Here, a peculiar problem arises in the use of Rf ; namely, we can calculate its value only for turbulent flow. We can use it to determine whether turbulent flow will become laminar, but not wheth laminar flow will become turbulent.

So, employing K-theory, $-\overline{w'\theta_v'} \propto \frac{\partial \overline{\theta_v}}{\partial z}$, ...

We can define the gradient Richardson number.

$$\boxed{R_i = \frac{\frac{g}{\overline{\theta_v}}\frac{\partial \overline{\theta_v}}{\partial z}}{\left[\left(\frac{\partial \overline{U}}{\partial z}\right)^2 + \left(\frac{\partial \overline{V}}{\partial z}\right)^2\right]}}$$

⇒ dynamical stability criteria

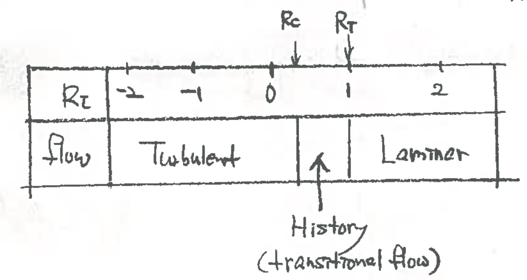
Critical Ri

Laminar flow becomes turbulent when $R_i < R_c$ ⟶ Usually 0.21 ~ 0.5

Turbulent flow becomes laminar when $R_i > R_T$ ← termination of turbulence

$R_T = 1$

∴



Basic PBL question

PBL (6480)

MET? (Physical question): Ruscher (1.5 hour, 1997) *

- Define the planetary boundary layer. Describe the evolution of the height of the boundary layer over a typical diurnal cycle, under synoptic scale anticyclone conditions. Describe the growth mechanisms for such a boundary layer, and include definitions for stable, neutral, and unstable boundary layers, and include a quantitative discussion of the importance of the Richardson number on boundary layer studies. Discuss the role of variations of cloud cover, wind speed, and soil moisture on boundary layer evolution.

Sol) → see my notes.

PBL : that part of the troposphere that is directly influenced by the presence of the earth's sfc and responds to sfc forcings with a timescale of about an hour or less.

⋮

Note for Isentrophic coordinates. (Ruscher)

- Advantages

① Over synoptic spatial + temporal scales, isentropic surfaces act like material surfaces ; for adiabatic processes, air parcel are thermodynamically bound to an isentropic sfc

② advection of pressure on θ sfc can be qualitatively determined by noting the cross isobar flow on θ sfc.

③ moisture transport on θ sfc includes the vertical advection component which is often missed in pressure coordinates.

④ narrow frontal bands are resolved in greater detail ← puts resolution where you want it.

⑤ Vertical gradient of isentropic sfc ("thickness") is a measure of static stability

⑥ the slope of an isentropic sfc in the vertical is directly related to the thermal wind.

⑦ parcel trajectories can be computed on θ sfc using implicit/explicit scheme

- Disadvantages

① θ sfcs are ill defined in regions where the lapse rate is neutral or superadiabatic

② continuity of θ sfc is disrupted significantly by the presence of diabatic effects.

③ θ sfcs can intersect the ground at large angles and thereby cause a problematic lower boundary condition

④ Choosing proper θ-sfc is not trivial.

⑤ Observations are not collected/ routinely processed on θ sfcs.

⑥ People are accustomed to the p-coordinate

Note for explosive cyclogenesis.

Explosive cyclogenesis is defined by Sanders and Gyakum (1980). It occurs when a low pressure system's central pressure decreases 1mb/hr for 24 hours at 60°N. This rate of decrease is known as a Bergeron.

Most case studies of explosive cyclogenesis have focused on storms that form over the western Atlantic and Pacific oceans.

"Maritime" and "land-based" bombs develop differently.

How does the latent heat play a role in the kinematics and energetics of explosive cyclogenesis? The maritime bomb is related to the largest sea surface temperature gradient.

Reed (1979) believes that CISK (Charney and Eliassen) may also play a major role in the formation and intensification of extratropical maritime bombs — as large role as baroclinic instability.

Latent heat release — another factor for an explanation of explosive cyclogenesis over land.

In the Sutcliffe - Pettersen development equation the heating term incorporating latent and diabatic heating effects helps describe the increase in circulation of a surface low. This is to say, the heating term is responsible for the generation of low level vorticity.

Petterssen (1956) emphasized that differential vorticity advection and the Laplacian of temperature advection are needed to initialize cyclogenesis, while latent and diabatic heating plays more of a role in sustaining and intensifying cyclogenesis.

In the area of the maximum rate of condensation, upward vertical motion increased as well as downward motion surrounding the precipitation area.

※ Latent heating influences the P field more slowly than ω field.
A change in the thickness pattern due to latent heating indicates a warm in the lower troposphere and a dynamic cooling in the upper troposphere.

In frontogenesis, the effect of latent heating is that low-level confluence of warm and cold air masses causes frontogenesis in the low-levels and set up a solenoidally direct vertical circulation about the upper cold front.

Latent heat release obviously has an important role in the kinematics as well as the energetics of extratropical cyclogenesis.

EAPE can be changed by

1) thermally direct circulation that converts eddy APE to eddy KE

2) Converting ZAPE to EAPE by destroying north-south temperature gradient

and 3) by creating east-west temperature gradient by differential diabatic heating

Danard (1964) investigated the effect of latent heating on the dynamics and energetics of a major winter cyclone in the central U.S. He concluded that the release of latent heat should lessen the tendency for vertical circulat to destroy existing temperature contrasts.

Bullock and Johnson (1971) found that the generation of APE is primarily dependent on the distribution of precipitation about the cyclone. They concluded this generation of APE compensates for the effects of friction within the storm.

→ Cyclogenesis over land

Gyakum + Barker (1988) claim that the initial phase of development is associated with mesoscale forcing. They state that the bulk diabatic effects of cumulus convection may directly affect incipient cyclogenesis.

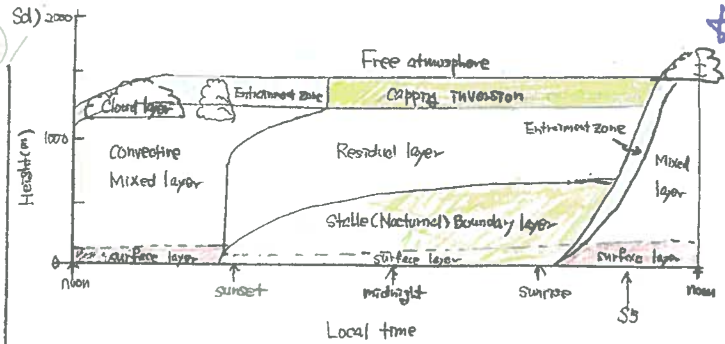
MET6480 PBL
Ruscher

(Spring 1997)

Boundary layer meteorology (Exam)

- Sketch the BL in high pressure regions over land.

Sol)

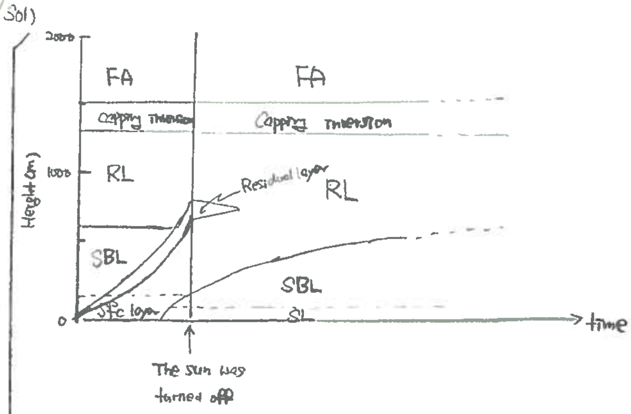


🟥 superadiabatic 🟩 strongly stable 🟦 weakly stable ☐ nearly adiabatic

The free atmosphere is at least weakly stable, but certainly not neutral. Don't forget, this is a layer, too! The entrainment zone is weakly stable, as well. The surface layer is only superadiabatic during the daytime over land, when surface heating can force the lapse rate near the ground to be as large as the autoconvective lapse rate (in a thin layer, this is 34.2K/km). At night over land, the SBL or NBL is stable, and becomes strongly stable over time. Over the warm ocean at night, it is possible to have neutral or perhaps even superadiabatic layers, but the problem asks for a land-based PBL. The other strongly stable layer is the capping inversion.

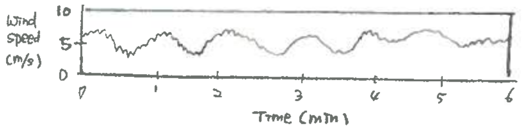
- Given the BL evolution, sketch the modified evolution that would occur if the sun was turned off at time S_s.

Sol)



Primary result of this cutting off the insolation in the morning is the retardation of growth of the daytime mixed layer. Result: a shallower daytime PBL for one thing, and likely (comparing to the original), less if any cloud development. Potential causes: clouds above PBL, forest fires, or other anthropogenic or natural pollution sources such as volcanic eruptions or chemical plumes, solar eclipse.

- If you observed the following measurement of wind speed on a strip chart, then what diameter (in meters) are the major eddies?



Sol) Define the time scale here as T (period of record) / n (# of eddies)

Then t = 360s/6.5 = 55.6s (approximately, but less than 60 seconds!)

The mean wind speed U is of order 6m/s (more than 5!). Assuming Taylor's hypothesis holds (eg, eddies move with the mean wind speed),

$L = t * U = \underline{335m}$

Suppose that the boundary layer can be approximated by a 1 km constant thickness layer over the whole continental United States. Above that, assume that a constant thickness (10 km) free atmosphere exists. Assume that each of these layers is well mixed for simplicity.

a) On the ~~average~~ average in spring, what fraction of the area of the continental United States is covered by cyclones (low pressure regions)?

Sol) According to most synoptic climatology references, approximately 25%–40% of the United States may be covered with cyclones during the spring season (i.e Petterssen, Roebber, Whitaker and Horn, Zishka and Smith)

Here, Lets use 1/3 for calculations below

b) What is the average vertical velocity magnitude out of the top of the BL in those cyclone region?

Sol) Low pressure systems ⇒ low level convergence (at PBL top, for example). If you integrate the continuity eq, you get $W(PBL) = -\nabla \cdot V * h$. Using appropriate values for a cyclone and h (given), $\nabla \cdot V = -1 \times 10^{-5} s^{-1}$ and $h = 1$ km, $W = 1 cm/s$. Values of order 1/2 to 5 or 10 cm/s are widely quoted for synoptic scale vertical motion estimates in the absence of deep convection.

c) What is the average subsidence velocity acting on the top of the BL in anticyclonic (high pressure) regions.?

Sol) Since low covers 1/3, $W(low) * A(low) + W(high) * A(high) = 0$ for mass conservation: Solve for W(high) gives −0.5 cm/s.

$$1 cm/s \, \frac{1}{3} + W(high) \frac{2}{3} = 0$$

What are the average residence time, P, of air within the BL and the free atmosphere? (Neglect horizontal advection out of the U.S.) $P = \frac{V}{F}$ ← volume ← (volume/time)

Sol) F (Volume flow rate) = Area * Velocity for each; area is horizontal value (whatever you pick or calculate), but it is the same for the PBL as it is for the FA, so it ends up cancelling out of the question. For the period, then, we have simply depth/vertical motion = Period.

For the PBL, $\boxed{P = \frac{h}{w}} = \frac{1000m}{10^{-2} m/s} = 10^5 s$ (~ 1 day)

→ seems about right... PBL recycles every day.

For the FA, $\boxed{P = \frac{H}{w}} = \frac{10000 m}{|-0.5 \times 10^{-2}|} = 2 \times 10^6 s$ ~ 23 days (seems large?)

The normal Rossby time scale cited by the same studies as above is from 5 to 7 days for an active springtime location, which is somewhat in between the two P above

Compare these residence times with the typical Rossby wave time period (that is, the average time period between cyclone passage over a fixed point). Comment on the significance of these times.

Sol) Relevance: The refresh rate for the FA is much larger than the Rossby time scale for cyclone passage, indicating that it is irrelevant for most real situations. The PBL turns over every day regardless of the synoptic situation. In cases where there are no cyclone passage (blocking situation) for many days, the FA can stagnate for long periods of time.

2/

- a) Given air at a pressure height of 90 kPa with a temperature of 30°C and a mixing ratio of 20g/kg, find the virtual potential temperature.

Sol) $P = 90\,kPa$, $T = 30°C = 303.16K$, $r = 20g/kg$ ← for unsaturated air

potential temperature (using Poisson's eq)

$$\theta = T\left(\frac{P_0}{P}\right)^{R/C_p} = 303.16 \times \left(\frac{100}{90}\right)^{0.286} \approx 312.43K$$

The air is unsaturated, so the virtual potential temp is

$$* \boxed{\theta_v = \theta(1+0.61r)}$$

$$= 312.43 \times [1+0.61 \times 0.02] \simeq 316.24K$$

b) Given saturated air at 850 hPa with a temperature of 20°C and a total water mixing ratio (i.e. sum of vapor and liquid mixing ratios) of 20 g/kg, find the virtual potential temp.

Sol) $P = 85\,kPa$, $T = 20°C = 293.16K$, $r_{tot} = r_{sat} + r_L = 20g/kg$ ← for saturated air

We might employ a ~~skew~~-T log-P diagram to find r_{sat} (saturation mixing ratio)

$\rightarrow r_{sat} \approx 17.8\,g/kg$.

So, $r_L = r_{tot} - r_{sat} = 20 - 17.8 = 2.2g/kg$

potential temp

$$\theta = T\left(\frac{P_0}{P}\right)^{R/C_p} = 293.16 \times \left(\frac{100}{85}\right)^{0.286} \simeq 307.11K$$

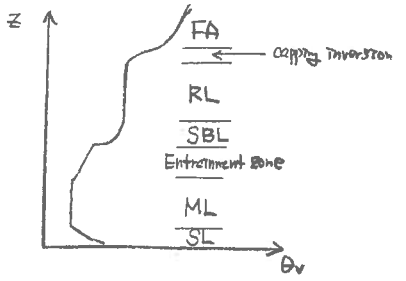
The air is saturated, so

$$\boxed{\theta_v = \theta(1+0.61r_{sat} - r_L)}$$

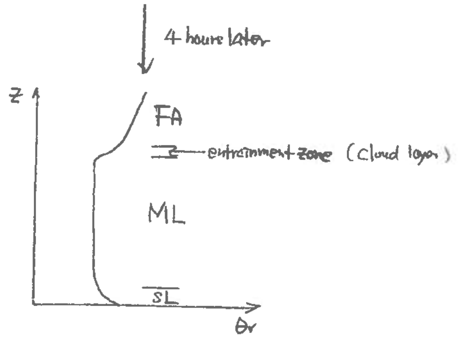
$$= 307.11 * (1+0.61 * 0.0178 - 0.0022) \simeq \underline{309.77K}$$

- Given the following virtual potential temp sounding, identify each layer. Estimate what time of day that sounding was made, and then sketch the θv profile that you might expect four hours later.

Sol)



In the morning after sunrise.



MET6480: Boundary layer Meteorology

Ruscher

- Download an 18 Z weather map for any day in April and print it out. It must have fronts and features analyzed on it, and it would be a good thing to have surface weather observations on it, as well. Suppose I am interested in learning which locations are appropriate to run the 1D PBL model for, to retrieve boundary layer forecasts for 12-24Z tomorrow. Discuss.

a) a set of 10 stations appropriate for running the model today
b) how to do the runs (from which soundings, any other difficulties you might have to deal with)?
c) Why we can't run the model everywhere for any time

- Today at 14 Z, the sky was overcast (not according to ASOS, which reported

KTLH 141352Z 03008KT 10SM CLR 12/04 A3015 RMK AO2 SLP208
T01220039=)

In any case, apparently, the clouds are at 12,000 ft. Winds are 8 knots, temperature is 54 F and the dew point is 39 F. How would you classify the stability of the boundary layer this morning, from this information, and any other information you care to use? Please define it using appropriate criteria!

- Please answer the following question from Stull: Problem 5.7.

- Please answer the following question from Stull: Problem 5.6 & 5.12 (counts as one problem).

- Please answer the following question from Stull: Problem 4.11

- Please answer the following question from Stull: Problem 3.19

- Describe the essential differences that must be taken into account when considering atmospheric boundary layer processes over the ocean (rather than the land). Consider the cases of both a warm sea surface and cool sea surface (relative to the air above it).

√4 Discuss how spectra of atmospheric measurements may be used to diagnose boundary layer and turbulence processes. Include a discussion of how to recognize dominant processes, dissipation, and the "spectral gap."

Boundary Layer meteorology (Ruscher)

- mean Structure of the PBL
- turbulent structure of the PBL
- governing prognostic eqs.
- Scaling techniques for PBL studies ; budget studies.
- Surface energy budget and radiative effects on PBL structure
- Similarity theory
- atmospheric convection + the convective boundary layer
- the stable boundary layer
- boundary layer cloud processes
- boundary layer modelling and turbulence closure
- Applications to air pollution meteorology.

notebook

⊙ flow in the free atmosphere (above PBL)

- gradient flow : horizontal frictionless flow, where the Coliolis acceleration and centripetal acceleration together balance the pressure gradient acceleration

$$\frac{V^2}{R} + fV = -g\frac{\partial Z}{\partial n} = -\frac{\partial \Phi}{\partial n}$$

where V : gradient wind speed

R : radius of curvature

- There are some special cases of gradient flow
- geostrophic flow : PGF + Coriolis forces balance

$$V_g = -\frac{g}{f}\frac{\partial Z}{\partial n}$$

- Cyclostrophic flow : PGF + centripetal force balance

$$\frac{V^2}{R} = -\frac{\partial \Phi}{\partial n}$$

- Inertial flow : frictionless flow on a geopotential surface on which there is no PGF. i.e.

$$\frac{V^2}{R} + fV = 0 \longrightarrow R = -\frac{V}{f}$$

⇒ one common characteristic : frictionless (inviscid)

⊙ Some properties of turbulence

- Turbulence is characterized by irregularity or randomness ; for practical purposes this aspect of turbulence must be treated statistically.
- Diffusivity : both passive + active properties are transported by turbulent motions in directions both parallel and normal to the mean flow. Turbulence generally transports properties "down the gradient"
- Turbulent flows are characterized by large Reynolds numbers

★ Re ≡ advective term(s)/viscous term(s)

$$= u_j\frac{\partial u_i}{\partial x_j} \Big/ \nu\frac{\partial^2 u_i}{\partial x_j^2} \approx U^2/L \Big/ \nu(U/L^2) = UL/\nu$$

→ Re > 1 indicates advective transports (including turbulent transports) exceed molecular transport (viscosity)

- Vorticity is a vector quantity once again (i.e. 3D) ; vortex stretching is necessary to maintain turbulence (vorticity is not just ζ in this class!)
- Dissipation of K.E. (conversion to turbulence energy)

Energy "cascades" down-scale, as energy is transfered from the larger scales to the smaller scales, ultimately leading to viscous dissipation

- A continuum can be assumed since the smallest scales of motion are larger than the mean free path (ok for the PBL + troposphere, not for the exosphere!)

→ This assumption is necessary to define thermal conductivity and viscosity.

- Turbulence is a feature of the dynamics of the flow and is not uniquely related to fluid properties such as viscosity.

⊙ methods of analysis for turbulence

- Dimensional analysis
  - variables can be scaled as O(1). In this way, one can formally eliminate many small scale terms in various eqs in the system of interest.

    eg $U\frac{\partial u}{\partial x} \approx U^2/L$: scale appropriately and then divide all terms in u-momen eg by U^2/L to get non-dimensional terms which are of same order and some which are smaller (OK if advective terms are dominant)
  - Solutions are a function of a limited number of scales.
  - The solutions may be functions of certain non-dimensional numbers (eg, Ra, Re, Ri, Ro, Ru, etc)

    ↳ Ruscher number

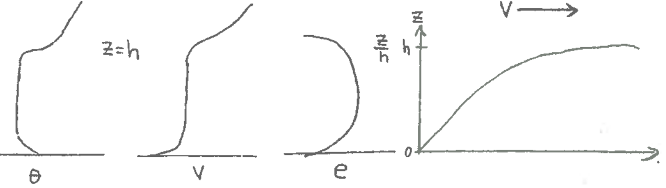
- Asymptotic invariance

The flow behavior may be well known in the limit as certain non-dimensional number approach zero or infinity. For example, as Re ⟶ ∞, the direct influence of viscosity on the flow becomes negligible and the flow approaches a state in which is independent of the Reynolds number (this is called asymptotic invariance). Any solution of turbulent flow should obey this condition as Re ⟶ ∞ ; that is, we now have a constraint on the flow. This is sometimes referred to in the literature as "Reynolds number similarity theory".

- Local invariance ☆

Whenever flows are not in balance, an adjustment process will occur which will be associated with a given time scale. If the adjustment time scale of the turbulence is small compared to the adjustment time scale of the external variables (basic state) or the mean flow, then the turbulence can adjust fast enough to be in equilibrium with the local mean flow and external variables. Then the turbulence is said to have "no memory" of the earlier conditions.

Exemple : convectively-driven mixed layers.



If the flow is sufficiently heated, the turbulence is generated primarily by buoyancy effects and may be described by a convective velocity scale.

$$\checkmark\ W_* \equiv \Big[\underbrace{g/\theta_0\,\overline{w'\theta'}}_{B}\,h\Big]^{1/3}$$

$\overline{w'\theta'}$ → kinematic vertical heat flux ($\frac{m}{s}K$)

B : measure of energy (energy/time) buoyancy

and a length scale, h. If the turbulence adjusts fast enough to be in equilibrium with W_* and h, then local turbulence is a function of W_* and h only. For example, we could postulate

$$e/W_*^2 = f(z/h) \Rightarrow \frac{e}{W_*^2} \text{ is nondimensional}$$

where $e \equiv$ turbulent K.E. $\equiv \frac{1}{2}[\overline{u'^2} + \overline{v'^2} + \overline{w'^2}]$, z is the height in the PBL, and h is the local value of the mixed layer depth. This type of ratio provides a normalization to the TKE as a function of normalized height within the PBL, an approach which characterizes many turbulence studies in the literature

At the mixed-layer interface itself, the turbulence is a function of the Richardson number, which we can define here as (being somewhat informal about it):

$$Ri = g/\theta_0 \, \Delta\theta \Delta h / (\Delta V)^2 \Rightarrow \frac{\frac{g}{\theta_0}\frac{\Delta\theta}{\Delta z}}{\left(\frac{\Delta V}{\Delta h}\right)^2}$$

where $\Delta\theta$ and ΔV are, respectively, the potential temperature and wind speed "jumps" across the interface of thickness Δh

Near the surface, shear generation may still dominate even under buoyant convection so that the turbulence locally (in the surface layer) may depend on a different velocity scale, called the friction velocity.

√ $u_* \equiv [\overline{u'w'}]^{1/2}$: empirical

$\longrightarrow$ upward flux of westerly momentum perturbation

as the turbulence is dominated primarily by shear effects

These are two examples of determination of velocity scales from the dominant dynamical processes for given flow situations

- The origin of turbulence

In the previous example, turbulence was generated by both buoyancy and shear. In the former case, heating generates convective currents + local shear which in turn generates turbulence.

Perturbation instability and generation of turbulence occurs if the local Reynolds number exceeds a critical value, which is typically over 1000. This value is lowest in the presence of disturbing mechanisms or a rough boundary.

Regardless of the value of the Reynolds number, turbulence is always dissipated by viscous effects. In stable stratification, such as the inversion interface of the previous example, buoyancy effects also destroy turbulence (here TKE is converted to P.E.)

- Under conditions of equilibrium, (where dimensional analysis and local invariance should be valid), production and destruction of turbulence is in balance.

For example, at the inversion interface, we might expect

Ri = ratio of buoyancy destruction : shear generation

to be fairly constant

- Summary of shear development of Turbulence

Richardson number

Reynolds number	$Ri < 0$	$0 < Ri < Ri_{critical}$	$Ri > Ri_{critical}$
$Re > Re_{critical}$	turbulence is generated by buoyancy effects if Ra is supercritical and generated mechanically	turbulence is generated mechanically	no turbulence
$Re < Re_{critical}$	turbulence is generated by buoyancy effects if Ra is supercritical	no turbulence	no turbulence

① Time scales

$p = \overline{p} + p' \Rightarrow$ important in momentum transport and redistribution

1. External time scales.

- Diurnal
- INERTIAL/Coriolis : $\frac{2\pi}{f} \sim 18\text{hr}$ in mid-lat.

Momentum balance $\left(\cancel{\frac{\partial u}{\partial t}}^{\,0} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial x}\right)$

$\hookrightarrow$ PBL : Ekman balance ; geostrophy + friction

$h \approx \sqrt{\frac{K}{f}}$ where K : eddy diffusivity (m^2s^{-1}) or mixing coef. or Austausch coef.

★ $\frac{\partial u}{\partial t} \propto \frac{\partial}{\partial z}\underset{\uparrow\, K=K(z)}{\textcircled{K}}\frac{\partial u}{\partial z} \quad (\propto \nabla_3^2 u)$

2. Velocity scale

V, U (large scale flows) : $\overline{u}, \overline{v}$

u, v (turbulent velocity scales) : u', v'

$u^2 = u_*^2 \underset{\uparrow \text{ parameterization}}{=} C_D U^2 \quad \because C_D \equiv$ drag coefficient. ($C_D = 7\times10^{-3}$ or 1.5×10^{-3})

$h \sim uT = C_D^{1/2} U T$

superimposed time scale (T); velocity scale (u); BL height (h)

h assumes near-neutrality ($\theta \sim$ const wrt z)

- Energy cascade $\Rightarrow$ energy moves from large scales $\longrightarrow$ smaller scales

$K \propto ul$

$\frac{U}{L} \sim \frac{W}{D}$ $\longrightarrow$ turbulence : $\left(\begin{matrix} u \approx w \\ l \approx d \end{matrix}\right)$

U: synoptic flow speed; W: vertical velocity; D: depth (of troposphere); L: synoptic length scale

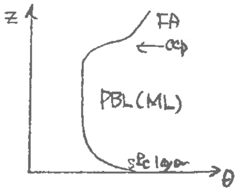
$U \gg W$, $L \gg D$

→ applications

a) sfc heat flux unimportant

$h \sim \frac{1}{4}\frac{u_*}{f}$

b) sfc heat flux is significant when ① well-mixed layer ② capping inversion



$h \propto W_*$: h is inversly proportional to stratification above the PBL $\left(\frac{\partial\theta}{\partial z}\right)$

c) SBL

$h \sim \frac{u_*}{N} \quad \because N$ = Brunt Vaisala frequency.

√ⓞ Surface energy balance (modelling aspects)

primitive eqs
- u, v
- w : hydrostatic
- p : state
- ρ : continuity of mass
- q : water vapor continuity ← interest.
- T, θ : thermodynamic energy ← interest.

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \underline{V}\cdot\nabla_h\theta + w\frac{\partial\theta}{\partial z} = \boxed{\dot{Q} + P - E}$$

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \underline{V}\cdot\nabla q + w\frac{\partial q}{\partial z} = \boxed{-\underset{\uparrow\,\text{precip.}}{P} + \underset{\uparrow\,\text{evap.}}{E}}$$

← concentrate

where $\dot{Q}$ = diabatic heating rate (K/s)

$\because$ conservation of heat $\dot{Q} = 0 \; (P = E) \Rightarrow \frac{d\theta}{dt} = 0$

$P, E \overset{\text{affect}}{\Longrightarrow} \theta, q$

$\hookrightarrow$ unit (W/m^2)

$L\downarrow - L\uparrow + S\downarrow - S\uparrow \cong 0$ (over long time, over the globe)

$\neq 0$ for us

(I) Another expression (K)

→ $R_n \neq 0$: net radiation can be partitioned

$I\downarrow = L\downarrow$ = atmospheric radiation (long-wave)

"greenhouse" active gases, CO_2, H_2O, O_3, NO_x, CFC

Clouds → 0 : no net contribution
1 : boost $L\downarrow$ by#

$L\uparrow$ = terrestrial radiation : Ice, water, soils, vegetation

Soils ← 12 types

water ← albedo

Ice ← Ice density, color, age, crystal characteristic

$\propto \varepsilon \sigma T^4$

emissivity $\varepsilon = \{0, \cdots, 1\}$

$K\downarrow = S\downarrow$ = direct + diffuse, scattering

↳ insolation (cloud, aerosol)

– Simple sun-earth geometry

θ, δ, h

$S\uparrow = \alpha S\downarrow$ (※ $S\downarrow - S\uparrow = (1-\alpha)S\downarrow$)

↳ albedo = % of visible light reflected from a surface.

※ 7% — UV
44% — VIS
49% — IR (near IR)

$R_n = ? = Q_n$ (radiation, energy)

$H = Q_H$ = sensible heat flux (convection)

$E = Q_E$ = latent heat flux (condensation, evaporation)

$G = Q_G$ = soil heat flux (conduction)

Positive flux ⇒ away from surface → atmosphere

ex) day time heating → sensible heating → $\theta\uparrow$

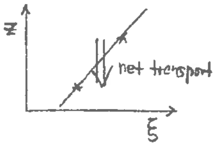
evaporation → increase q ($\theta\downarrow$)

negative flux

⊙ Friction ⟺ turbulence

Simple systems

1) Gradients : $\frac{\partial \xi}{\partial t} \propto \frac{\partial \xi}{\partial z}$ (ξ: any variable)



2) $\frac{\partial \xi}{\partial t} \propto \frac{\partial}{\partial z}\left(K_\xi \frac{\partial \xi}{\partial z}\right)$ → $f(z)$

3) $\frac{\partial \xi}{\partial t} \propto K \frac{\partial^2 \xi}{\partial z^2}$

↑ K = const

⊙ Richardson

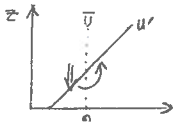
· gradient $R_i = \dfrac{g/\bar{\theta}\,(\partial\bar{\theta}/\partial z)}{\left(\frac{\partial U}{\partial z}\right)^2}$

· flux $R_i = \dfrac{g/\theta_0\ \overline{w'\theta'}}{-\overline{u'w'}\left(\frac{\partial U}{\partial z}\right)}$

∵ $\overline{w'\theta'}$ = kinematic heat flux

$\overline{w'\theta'} > 0$ ⇒ positive heat flux

$\overline{u'w'}$ = kinematic momentum flux



normally, $\overline{u'w'} < 0$ for $\frac{\partial \bar{\theta}}{\partial z} > 0$

$= \frac{K_h}{K_m}(R_i) = P_r^{-1} R_i$ ∵ $P_r \sim 0.74 \approx 1$ (R_i: gradient R_i)

· Ri_B (Bulk R_i) $= R_i$ where $Z_b = 0$ (bottom)

$\Delta Z = Z_t - Z_b$

simplify $\frac{\partial}{\partial z}$ as $U \longrightarrow 0$ at $z=0$

· Ri_c = critical R_i, $1/4 < Ri_c < 1$

⊙ Reynolds averaging

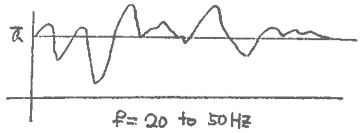
$a = \bar{a} + a'$

$b = \bar{b} + b'$

$\overline{a'} = \overline{b'} = 0$

$\overline{a'^2} \neq 0$

$\overline{a'b'} \neq 0$



f = 20 to 50Hz

1Hz = 1s⁻¹

$a'(t) = a - \bar{a}$

$\overline{ab} = \overline{(\bar{a}+a')(\bar{b}+b')}$ ∵ mean is a linear operator

$= \overline{\bar{a}\bar{b}} + \overline{\bar{a}b'} + \overline{a'\bar{b}} + \overline{a'b'}$

$= \bar{a}\bar{b} + \cancel{\bar{a}\overline{b'}}^{0} + \cancel{\overline{a'}\bar{b}}^{0} + \overline{a'b'} = \bar{a}\bar{b} + \overline{a'b'}$

ex) heat flux : $\overline{w'\theta'}$, $\bar{w}\bar{\theta}$

$\bar{\theta} = 300K$
$|\bar{w}| = 10^{-2}\,ms^{-1}$
$|w'| = 1\,ms^{-1}$
$|\theta'| = 3K$
→ $|\bar{w}\bar{\theta}| \sim 3K\frac{m}{s}$
$|\overline{w'\theta'}| \sim 3K\frac{m}{s}$

$\frac{\partial \theta}{\partial t} \sim \frac{\partial}{\partial z}$(heat flux)

⊙ Statistic

$var(A) \equiv \sigma_A^2 = \overline{a'^2}$

$\sigma_A \equiv \sqrt{\overline{a'^2}}$

turbulence intensity $\boxed{I_a = \frac{\sigma_A}{\bar{A}}}$ $\left(= \frac{\sigma_u}{\bar{U}}\right)$ (ex)

$covar(a,b) = \frac{1}{N}\sum_{i=1}^{N} a_i' b_i' = \overline{a'b'}$

$r_{ab} = \dfrac{covar(a,b)}{\sigma_a \sigma_b}$

⊙ Friction, fluxes + stress

$\frac{\partial u}{\partial t} = \cdots, (F_u) \xRightarrow[\text{parameterization}]{} F_u \propto \frac{\partial}{\partial z} K_n \frac{\partial u}{\partial z}$

$F_u \propto \frac{\partial}{\partial z}\overline{u'w'}$

$F_u \propto \frac{1}{\rho}\frac{\partial}{\partial z}\tau_s$

$\tau_{xz} = -\rho\,\overline{u'w'}_s = \tau_{zx} = -\rho\,\overline{w'u'}_s$

⇒ $\overline{u'w'} = \overline{w'u'}$

$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_B - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\delta_{ij}$

Viscous shear stresses

bulk viscosity $= \mu_B$, $\nu = \frac{\mu}{\rho}$ = kinematic viscosity.

eddy viscosity K

$K \gg \nu$

K ~ 1 ~ 100 m²/s, ν ~ 10⁻⁵ m²/s

※ Reynolds stresses ($\overline{u'w'}$) ≫ viscous shear stresses.

⊙ mean flow eqs

$$u=\overline{u}+u' \;,\; v=\overline{v}+v' \;.\; w=\overline{w}+w' \quad \text{———} ①$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{———} ②$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{———} ③$$

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \quad \text{———} ④$$

$\rho * ② \Rightarrow$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}-\rho fv=-\frac{\partial p}{\partial x}$$

$u * ④ \Rightarrow$

$$+\;\; \underline{u\frac{\partial \rho}{\partial t}}+u\frac{\partial}{\partial x}(\rho u)+u\frac{\partial}{\partial y}(\rho v)+u\frac{\partial}{\partial z}(\rho w)=0$$

$\Rightarrow$

$$\boxed{\rho\frac{\partial u}{\partial t}}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}-\rho fv+\boxed{u\frac{\partial \rho}{\partial t}}+u\frac{\partial}{\partial x}(\rho u)+u\frac{\partial}{\partial y}(\rho v)$$
$$+u\frac{\partial}{\partial z}(\rho w)=-\frac{\partial p}{\partial x}$$

$\Rightarrow$

$$\boxed{\frac{\partial}{\partial t}(\rho u)}+\frac{\partial}{\partial x}(\rho u^2)+\frac{\partial}{\partial y}(\rho uv)+\frac{\partial}{\partial z}(\rho uw)-\rho fv=-\frac{\partial p}{\partial x} \quad \text{———} Ⓐ$$

Similarly

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho v^2)+\frac{\partial}{\partial z}(\rho vw)+\rho fu=-\frac{\partial p}{\partial y} \quad \text{———} Ⓑ$$

Take the ensemble mean of eqs Ⓐ + Ⓑ and make the following assumptions

$$\overline{\frac{\partial}{\partial t}}=\frac{\partial}{\partial t}\;,\; \rho=\rho'+\overline{\rho}\;.\; \rho'\ll\overline{\rho}\rightarrow\rho\approx\overline{\rho}\;(\text{no } \overline{\rho' u'} \text{ term}),$$

∴

$$\overline{\rho u}=\overline{\rho(\overline{u}+u')}=\overline{\rho u}+\overline{\rho u'}=\overline{\rho}\,\overline{u}$$

$$\overline{\rho u^2}=\overline{\rho(\overline{u}+u')(\overline{u}+u')}=\overline{\rho}\,\overline{u}\,\overline{u}+\overline{\rho}(\overline{u'u'})$$

$$\overline{\rho uv}=\overline{\rho(\overline{u}+u')(\overline{v}+v')}=\overline{\rho}\,\overline{u}\,\overline{v}+\overline{\rho}(\overline{u'v'})$$

$$\overline{\rho uw}=\overline{\rho(\overline{u}+u')(\overline{w}+w')}=\overline{\rho}\,\overline{u}\,\overline{w}+\overline{\rho}(\overline{u'w'})$$

$$\overline{\rho fv}=\overline{\rho}f\overline{v}$$

$$\overline{\frac{\partial p}{\partial x}}=\frac{\partial\overline{(\overline{p}+p')}}{\partial x}=\frac{\partial\overline{p}}{\partial x}$$

Ⓐ →

$$\frac{\partial}{\partial t}(\overline{\rho}\,\overline{u})+\frac{\partial}{\partial x}(\overline{\rho}\,\overline{u}\,\overline{u})+\frac{\partial}{\partial x}(\overline{\rho}\,\overline{u'u'})+\frac{\partial}{\partial y}(\overline{\rho}\,\overline{u}\,\overline{v})+\frac{\partial}{\partial y}(\overline{\rho}\,\overline{u'v'})$$
$$+\frac{\partial}{\partial z}(\overline{\rho}\,\overline{u}\,\overline{w})+\frac{\partial}{\partial z}(\overline{\rho}\,\overline{u'w'})-f\overline{\rho}\,\overline{v}=-\frac{\partial\overline{p}}{\partial x}$$

$\overline{u} * \overline{④} \Rightarrow$

∴

$$\overline{\overline{u}\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]}=0$$

$$\overline{u}\frac{\partial\overline{\rho}}{\partial t}+\overline{u}\frac{\partial}{\partial x}(\overline{\rho}\,\overline{u})+\overline{u}\frac{\partial}{\partial y}(\overline{\rho}\,\overline{v})+\overline{u}\frac{\partial}{\partial z}(\overline{\rho}\,\overline{w})=0$$

So,

$$\overline{\rho}\frac{\partial\overline{u}}{\partial t}+\overline{\rho}\,\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{\rho}\,\overline{w}\frac{\partial\overline{u}}{\partial z}-f\overline{\rho}\,\overline{v}$$
$$=-\frac{\partial\overline{p}}{\partial x}-\left[\frac{\partial}{\partial x}(\overline{\rho}\,\overline{u'u'})+\frac{\partial}{\partial y}(\overline{\rho}\,\overline{u'v'})+\frac{\partial}{\partial z}(\overline{\rho}\,\overline{u'w'})\right]$$

⇒ divide by $\overline{\rho}$

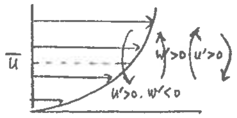
$$\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z}-f\overline{v}=-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x}-\frac{1}{\overline{\rho}}\left[\qquad \text{''} \qquad\right]$$

Lets define the viscous stress

$$\tau^*_{ij}=\begin{bmatrix}\tau^*_{xx} & \tau^*_{xy} & \tau^*_{xz}\\ \tau^*_{yx} & \tau^*_{yy} & \tau^*_{yz}\\ \tau^*_{zx} & \tau^*_{zy} & \tau^*_{zz}\end{bmatrix} \qquad \tau_{ij}=\tau_{ji} \text{ for } i\neq j$$

⇒ Reynolds stress tensor

$$\tau^*_{ij}=\begin{bmatrix}-\rho\overline{u'u'} & -\rho\overline{u'v'} & -\rho\overline{u'w'}\\ -\rho\overline{v'u'} & -\rho\overline{v'v'} & -\rho\overline{v'w'}\\ -\rho\overline{w'u'} & -\rho\overline{w'v'} & -\rho\overline{w'w'}\end{bmatrix}$$



$\overline{u'w'}<0$ (downward) ?

∴ final eqs by using the Reynolds stress (drop ¯)

$$\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv+\frac{1}{\rho}\left[\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right]$$

$$\frac{dv}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu+\frac{1}{\rho}\left[\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right]$$

⊙ Eddy viscosity (= mixing coefficient, Austausch coefficient)

$$\tau=\mu\frac{\partial V}{\partial z} \quad \text{and} \quad \tau^*=K_m\frac{\partial\overline{V}}{\partial z}$$

$$\Rightarrow\frac{\partial u}{\partial t}\sim\frac{1}{\rho}\frac{\partial}{\partial z}K_m\frac{\partial u}{\partial z}$$

↳ Shows how strong the turbulence is.
($K_m=K_m(z)$)

⊙ Surface layer

In general, the surface layer involves 10% of the PBL.

In the Surface layer $\tau\neq\tau(z)$ while $\tau=\tau(z)$ in the PBL

Constant flux or Constant stress layer → Bad!

→ $C_m, C_h=f(z)$

↳ exchange coefficient for surface layer

⊙ Ekman layer

assumptions for Ekman layer

① Horizontal mean motion is unaccelerated. i.e $\frac{\partial V}{\partial t}=0$

② $\frac{\partial\overline{u}}{\partial x}, \frac{\partial\overline{v}}{\partial y}$ are small compared to vertical wind shear $\left(\frac{\partial\overline{u}}{\partial z}, \frac{\partial\overline{v}}{\partial z}\right)$

③ At all levels, force balance is Coriolis force; PGF, eddy viscous effects.

$$0=-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x}+f\overline{v}+\frac{1}{\overline{\rho}}\left[\frac{\partial\tau_{xz}}{\partial z}\right]$$

$$0=-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial y}-f\overline{u}+\frac{1}{\overline{\rho}}\left[\frac{\partial\tau_{yz}}{\partial z}\right]$$

④ Eddy viscosity K_m is independent of height. $K_m\neq K_m(z)$

⑤ The large scale pressure gradient + density do not vary significantly with height and may be considered constant

$$\tau_{zx}=\tau_{xz}=\overline{\rho}K_m\frac{\partial\overline{u}}{\partial z} \quad \text{and} \quad \tau_{zy}=\tau_{yz}=\overline{\rho}K_m\frac{\partial\overline{v}}{\partial z}$$

So

$$0=-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x}+f\overline{v}+\left[K_m\frac{\partial^2\overline{u}}{\partial z^2}\right]$$

$$0=-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial y}-f\overline{u}+\left[K_m\frac{\partial^2\overline{v}}{\partial z^2}\right]$$

⑥ $\frac{\partial V_g}{\partial z}=0. \rightarrow \overline{u_g}=-\frac{1}{\rho f}\frac{\partial\overline{p}}{\partial y}$ and $\overline{v_g}=\frac{1}{\rho f}\frac{\partial\overline{p}}{\partial x}$

$$0=f\overline{v}-f\overline{v_g}+\left[K_m\frac{\partial^2\overline{u}}{\partial z^2}\right] \quad \text{———} ⓐ$$

$$0=-f\overline{u}+f\overline{u_g}+\left[K_m\frac{\partial^2\overline{v}}{\partial z^2}\right] \quad \text{———} ⓑ$$

⇒ Note $V=\overline{u}+i\overline{v}$, $i=\sqrt{-1}$

ⓐ + ⓑ i ⇒

$$0=-if\overline{u}+if\overline{u_g}+f\overline{v}-f\overline{v_g}+\left[K_m\frac{\partial^2(\overline{u}+i\overline{v})}{\partial z^2}\right]$$

$$\Rightarrow if(\overline{u_g}+i\overline{v_g})-if(\overline{u}+i\overline{v})+K_m\frac{\partial^2(\overline{u}+i\overline{v})}{\partial z^2}=0$$

$$\Rightarrow +ifV_g-ifV+K_m\frac{\partial^2 V}{\partial z^2}=0$$

$$\Rightarrow -if(V-V_g)+K_m\frac{\partial^2 V}{\partial z^2}-\overset{0}{K_m\frac{\partial^2 V_g}{\partial z^2}}=0$$

$$\Rightarrow -if(V-V_g)+K_m\frac{\partial^2(V-V_g)}{\partial z^2}=0$$

Let $W=V-V_g$: ageostrophic wind

$$-ifW+K_m\frac{\partial^2 W}{\partial z^2}=0$$

$$\frac{\partial^2 W}{\partial z^2}-\frac{if}{K_m}W=0$$

Let $i=\frac{1}{2}(i+1)^2$ and $m^2=\frac{f}{2K_m}$

$$\Rightarrow \frac{\partial^2 W}{\partial Z^2} - (i+1)^2 m^2 W = 0$$

↓ solution

$$W = ae^{(i+1)mz} + be^{-(i+1)mz}$$

B.C ①

$Z \to \infty$, $V \to V_g$ and $W = 0$ at $Z \to \infty$

$0 = ae^{(i+1)m\infty} + be^{-(i+1)m\infty}$

Since $e^{\infty} \to \infty$, "a" must be zero.

$W = be^{-(i+1)m\infty}$

B.C ②

$Z \to 0$, $V \to 0$ and $W = -V_g$ at $Z = 0$

$-V_g = be^{-(i+1)m0}$ and so $W = -V_g e^{-(i+1)mz}$

$\Rightarrow$ $W = V - V_g = -V_g e^{-mz}(\cos mz - i\sin mz)$

Since $V = \bar{u} + i\bar{v}$ + $V_g = \bar{u}_g + i\bar{v}_g$

$$(\bar{U} + i\bar{V}) - (\bar{u}_g + i\bar{v}_g) = -(\bar{u}_g + i\bar{v}_g)e^{-mz}(\cos mz - i\sin mz)$$
$$= e^{-mz}(-\bar{u}_g\cos mz + i\bar{u}_g\sin mz - i\bar{v}_g\cos mz - \bar{v}_g\sin mz)$$

∴ the Ekman solution

$$\bar{U} = \bar{U}_g(1 - e^{-mz}\cos mz) - \bar{V}_g e^{-mz}\sin mz$$
$$\bar{V} = \bar{V}_g(1 - e^{-mz}\cos mz) + \bar{U}_g e^{-mz}\sin mz$$

If $\bar{V}_g = 0$.

$$\bar{U} = \bar{U}_g(1 - e^{-mz}\cos mz)$$
$$\bar{V} = \bar{U}_g e^{-mz}\sin mz$$

$\Rightarrow$ The classical Ekman Spirals solution

$$\frac{\bar{U}}{\bar{U}_g} = (1 - e^{-mz}\cos mz)$$
$$\frac{\bar{V}}{\bar{U}_g} = e^{-mz}\sin mz$$

∴ If veering ⟶ warm advection.

⊙ Turbulent eqs

positive or negative due to ordering of i,j

$$\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_j\frac{\partial \bar{U}_i}{\partial x_j} = -\delta_{i3}g + f\epsilon_{ij3}\bar{U}_j - \frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial x_i} + \nu\frac{\partial^2 \bar{U}_i}{\partial x_j^2} - \frac{\partial}{\partial x_j}\overline{(u_i'u_j')}$$

↑ must close

$P = \rho R T_v$

$\bar{P} + p' = R(\bar{\rho} + \rho')(\bar{T}_v + T_v')$

$= R(\bar{\rho}\bar{T}_v + \bar{\rho}T_v' + \rho'\bar{T}_v + \rho' T_v')$

Reynolds averaging

$\bar{P} = R(\bar{\rho}\bar{T}_v + \overline{\rho' T_v'})$ ∵ $\bar{\rho}\bar{T}_v \gg \overline{\rho' T_v'}$

$P' = R(\bar{\rho}T_v' + \rho'\bar{T}_v + \rho' T_v')$

$$\hookrightarrow \frac{P'}{\bar{P}} = \frac{\rho'}{\bar{\rho}} + \frac{T_v'}{\bar{T}_v} + \frac{\rho' T_v'}{\bar{\rho}\bar{T}_v}$$

⋮

look at Stull's book

END Note book

Stull's book

Ch1. mean boundary layer characteristics.

1.1 A boundary-layer definition

We can define the boundary layer as that part of the troposphere that is directly influenced by the presence of the earth's surface and responds to surface forcings with a timescale of about an hour or less.

→ surface forcings: ① frictional drag, ③ evaporation ② transpiration ④ heat transfer, ⑤ pollutant emission + ⑥ terrain induced flow modifi

→ diurnal variation : one of the key characteristics of the BL over land

→ Turbulence is one of the important transport processes.

fair-weather cumulus cloud
stratocumulus cloud (fog)

1.2 Wind and flow

- wind: mean wind, turbulence, waves → each can exist in the BL.

→ transport of quantities (eg moisture, heat, momentum + pollutants) is dominated in the horizontal by the mean wind ✓; in the vertical by turbulence ✓

→ waves (usually nighttime BL)
transport little heat, humidity, and other scalars such as pollutants.
However, effective at transporting momentum and energy

1.3 Turbulent transport.

- The relative strengths of different scale eddies define the turbulence spectrum
- Turbulence, the gustiness superimposed on the mean wind, can be visualized as consisting of irregular swirls of motion called eddies.
- BL turbulence by forcings from ground
  - solar heating → thermals
  - frictional drag → wind shears
  - obstacles → turbulent wakes
- Turbulence is several orders of magnitude more effective at transporting quantities than is molecular diffusivity.

1.4 Taylor's hypothesis

- Taylor (1938) suggested that for some special cases, turbulence might be considered to be frozen as it advects past a sensor.
- Taylor's simplification is useful for only those cases where the turbulent eddies evolve with a timescale longer than the time it takes the eddy to be advected past a senso
- M (total wind magnitude) ⟶ $M^2 = U^2 + V^2$

✓ $P = \lambda / M$ (λ: eddy diameter; P: time period)

- For any variable ξ, Taylor's hypothesis states that turbulence is frozen when $\frac{d\xi}{dt} = 0$ ⇒ the general form of Taylor's hypothesis

$$\frac{\partial \xi}{\partial t} = -U\frac{\partial \xi}{\partial x} - V\frac{\partial \xi}{\partial y} - W\frac{\partial \xi}{\partial z}$$

- using wavenumber K + frequency f

✓ $K = \frac{f}{M} \longrightarrow \frac{2\pi}{P}$

$\llcorner \frac{2\pi}{\lambda}$

- Condition : To satisfy ~~the~~ requirements that eddy have negligible change as it advects past a sensor

$\sigma_M < 0.5M$

* standard deviation ⟹ a measure of the

- Taylor's hypothesis should be satisfactory when the turbulence intensity is small relative to the mean wind speed.

1.5 Virtual potential temperature

- Buoyancy is one of the driving forces for turbulence in the BL.
  - ↳ thermals : positively buoyant
- Virtual temperature : temperature that dry air must have to equal the density of moist air at the same pressure
  - ↳ instead of density
- Water vapor is less dense than dry air.
  - → moist unsaturated air is more buoyant than dry air of the same T

※ $\rho\downarrow \rightarrow \theta_v\uparrow$

$\rho\uparrow \rightarrow \theta_v\downarrow$

- definition

For saturated (cloudy) air, $\theta_v = \theta(1+0.61r_{sat} - r_L)$

liquid-water mixing ratio

water-vapor saturation mixing ratio

For unsaturated air, $\theta_v = \theta(1+0.61r)$

$$\theta = T\left(\frac{P_o}{P}\right)^{0.286}$$

1.6 Boundary layer depth and Structure

- Over oceans, the boundary layer depth varies relatively slowly in space + time (due to a slowly varying SST)

✓ The general nature of the BL is to be thinner in high-pressure regions than in low-pressure regions.

- In low pressure regions, cloud base is often used as an arbitrary cut-off for BL studies.

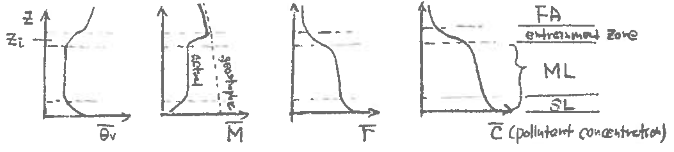
✱ look at the figure of boundary layer structure. (in exam)

- surface layer : the region at the bottom of the boundary layer where turbulent fluxes + stress vary by less than 10% of their magnitude

- BL : boundary layer = PBL = ABL
  - CL : Cloud layer
  - FA : Free atmosphere
  - IBL : Internal boundary layer
  - ML : mixed layer = CBL (convective boundary layer)
  - RL : Residual layer
  - SBL : Stable boundary layer = NBL (nocturnal BL)
  - SCL : Subcloud layer
  - SL : surface layer (the bottom 10% of the BL)
- more notations
  - h : Top of the SBL
  - Z_i : Top of the ML
  - Z_r : Top of the RL
  - Z_b : Top of SCL ~ LCL (lifting condensation level)

(1) Mixed Layer

- Turbulence is usually convectively driven.
  - → convective sources : heat transfer from a warm ground surface
  - radiative cooling from the top of the CL.
- Even when convection is the dominant mechanism, there is usually wind shear across the top of the ML ⟹ turbulence generation (Kelvin-Helmholtz waves)
- looping
- Typical daytime profiles



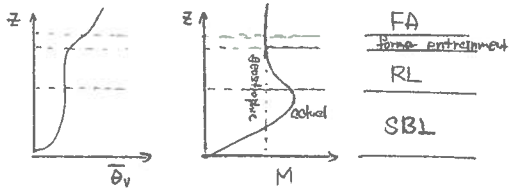
- The entrainment zone is frequently called an inversion layer
- Wind speeds are subgeostrophic throughout the ML.

(2) Residual layer

- neutrally stratified.
- Coning (cone-shaped plume)

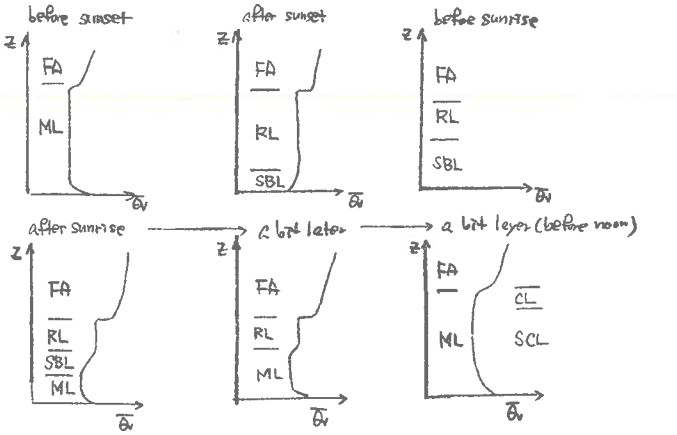
(3) Stable Boundary Layer

- The statically stable air tends to suppress turbulence, while the developing nocturnal jet enhances wind shears that tend to generate turbulence.
- poorly defined top.
- fanning
- Winds exhibit a very complex behavior at night
- An idealized stable boundary layer in a high-pressure region profiles



- wave motions are a frequent occurrence in the SBL
  - → The strongly stable NBL not only supports gravity waves, but it can trap many of the higher-frequency waves near the ground.

(4) θ_v evolution



- Knowledge of the θ_v profile is usually sufficient to identify the parts of the BL.
- lofting + fumigation

in mixed layer (night plume + new plume)

1.7 micrometeorology

- Stochastic methods : the average statistical effects of the eddies.
- Similarity theory : the apparent common-behavior exhibited by many empirically-observed phenomena, when properly scaled.
- phenomenological classifications : the largest size structures such as thermals are classified and sometimes approached in a partially deterministic manner.
- air-pollution meteorology + agricultural meteorology.

1.8 Significance of the BL

- Comparison of BL + FA characteristics.

Property	BL	FA
Turbulence	almost continuously turbulent over its whole depth.	turbulence in convective clouds, and sporadic CAT in thin layers of large horizontal extent.
Friction	Strong drag against the earth's surface. Large energy dissipation	Small viscous dissipation

Dispersion	Rapid turbulent mixing in the vertical and horizontal	Small molecular diffusion, often rapid horizontal transport by mean wind
winds	near logarithmic wind speed profile in the surface layer. Subgeostrophic, cross-isobaric flow common	winds nearly geostrophic
vertical transport	turbulence dominates.	Mean wind and cumulus-scale dominate.
thickness	Varies between 100 m to 3km in time and space. Diurnal oscillations over land	Less variable 8–18km Slow time variations.

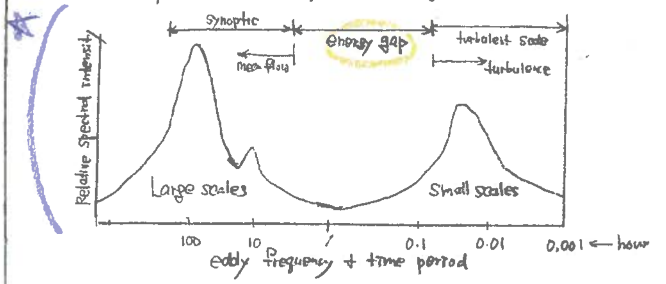
Ch2. Some mathematical + conceptual tools : Statistics

2.0

- The randomness of turbulence makes deterministic description difficult
  → Statistics.
- Variances : measures of turbulence intensity or TKE.
- Covariances : measures of flux or stress.

2.1 The signature of turbulence and its spectrum.

- If we look at a anemonograph (wind speed measurement), we can find,
  → wind speed varies in an irregular pattern
  → The ability to find a statistically-stable mean value suggests that turbulence is not completely random
  → we can use statistics such as the variance or standard deviation to characterize the turbulence intensity.
  → evidence of the spectrum of turbulence
- Schematic spectrum of wind speed near the ground



- Energy cascade : a net transfer of turbulence energy from the larger to the smaller eddies

2.2 The spectral gap

- In the microscale sense, the spectral gap provides a means to seperate the turbulent from the nonturbulent influences on the BL

2.3 Mean and turbulent parts.

$u' = U - \overline{U}$

- The existence of a spectral gap allows us to partition the flow field like above
- partition each of variables into mean + turbulent parts.

$$\begin{cases} U = \overline{U} + u' \\ V = \overline{V} + v' \\ W = \overline{W} + w' \\ \theta_v = \overline{\theta_v} + \theta_v' \\ 8 = \overline{8} + 8' \\ c = \overline{c} + c' \end{cases}$$

2.4 Some basic statistical methods

(1) The mean

- time average

$${}^{t}\overline{A(s)} = \frac{1}{N}\sum_{i=0}^{N-1} A(i,s) = \frac{1}{P}\int_{t=0}^{P} A(t,s)\,dt$$

time period

(statistically the same at every point in space)

- Spatial average → if the turbulence is homogeneous → make a spatial average meaningful

$${}^{s}\overline{A}(t) = \frac{1}{N}\sum_{j=0}^{N-1} A(t,j) = \frac{1}{S}\int_{t=0}^{S} A(t,s)\,ds$$

- ensemble average → sum over N identical experiments

$${}^{e}\overline{A}(t,s) = \frac{1}{N}\sum_{i=0}^{N-1} A_i(t,s)$$

- Spatial average → Volume, Area, Line averages.
- For turbulence that is both homogeneous and stationary (statistically not changing over time), the time, space, and ensemble averages should all be equal.
  ⇒ ergodic condition ✓

(2) Rules of averaging

- $\overline{A+B} = \overline{A} + \overline{B}$, $\overline{cA} = c\overline{A}$, $\overline{\overline{A}} = \overline{A}$, $\overline{\overline{A}B} = \overline{A}\,\overline{B}$
- $\overline{\frac{dA}{dt}} = \frac{\partial \overline{A}}{\partial t}$, $\overline{\left(\frac{dA}{dt}\right)} = \frac{d\overline{A}}{dt}$

(3) Reynolds averaging

$\overline{A} = \overline{\overline{A}+a'} = \overline{A} + \overline{a'} \longrightarrow \overline{a'} = 0$

$\overline{A \cdot B} = \overline{(\overline{A}+a')(\overline{B}+b')} = \overline{\overline{A}\,\overline{B} + a'\overline{B} + \overline{A}b' + a'b'} = \overline{A}\,\overline{B} + \overline{a'b'}$

(4) Variance, standard deviation and turbulent intensity

$\sigma_A^2 = \frac{1}{N}\sum_{i=0}^{N-1}(A_i - \overline{A})^2 \longrightarrow$ biased variance → usually used

$\sigma_A^2 = \frac{1}{N-1}\sum_{i=0}^{N-1}(A_i - \overline{A})^2 \longrightarrow$ unbiased variance

$\because a' = A - \overline{A}$

$\sigma_A^2 = \frac{1}{N}\sum_{i=0}^{N-1} a_i'^2 = \overline{a'^2}$

$\sigma_A = (\overline{a'^2})^{1/2} \longrightarrow$ Standard deviation.

- a dimensionless measure of the turbulence intensity (I)

$I = \sigma_M / \overline{M}$ ← mean wind speed

→ I<0.5 is required for Taylors hypothesis to be valid. ✓

(5) Covariance and correlation

- $\mathrm{covar}(A,B) \equiv \frac{1}{N}\sum_{i=0}^{N-1}(A_i - \overline{A})(B_i - \overline{B})$

  $= \overline{a'b'}$ → the degree of common relationship between the two varia

- linear correlation coefficient (r_{AB}) ← a normalized covariance

$$r_{AB} \equiv \frac{\overline{a'b'}}{\sigma_A \sigma_B}$$

2.5 Turbulent kinetic energy. (TKE)

- $KE = 0.5\, m M^2$ (↳ mass) ⟶ $KE/_{\text{unit mass}} = 0.5 M^2$

$$\begin{cases} \frac{\overset{\text{mean}}{MKE}}{m} = \frac{1}{2}(\overline{U}^2 + \overline{V}^2 + \overline{W}^2) \\ e = \frac{1}{2}(u'^2 + v'^2 + w'^2) \end{cases}$$

↳ an instantaneous TKE per unit mass

$$\overline{e} = \frac{TKE}{m} = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$$

2.6 Kinematic flux

- flux : the trasfer of a quantity "per unit area per unit time"

Flux	Symbol	Units
mass	$\tilde{M}$	$\left[\frac{kg_{air}}{m^2\cdot s}\right]$
heat	$\tilde{Q}_H$	$\left[\frac{J}{m^2 s}\right] = \frac{W}{m^2}$
moisture	$\tilde{R}$	$\left[\frac{kg_{water}}{m^2 s}\right] \rightarrow \tilde{Q}_E = L_v \tilde{R}$ (latent heat flux; L_v: J/kg)
momentum	$\tilde{F}$	$\left[\frac{kg(m\cdot s^{-1})}{m^2 s}\right] = \frac{N}{m^2}$ → units for stress
pollutant	$\tilde{\chi}$	$\left[\frac{kg_{pollutant}}{m^2 s}\right]$ or $\left[\frac{kg_{pollutant}}{m^3}\frac{m}{s}\right]$

- Unfortunately, we rarely measure quantities such as heat or momentum directly. Instead we measure things like temperature or wind speed.

Kinematic flux	Symbol	Equation	Units
mass	M	$= \frac{\tilde{M}}{\rho_{air}}$	$\left[\frac{m}{s}\right]$
heat	Q_H	$= \frac{\tilde{Q}}{\rho_{air} C_p}$	$\left[K\frac{m}{s}\right]$
moisture	R	$= \frac{\tilde{R}}{\rho_{air}}$	$\left[\frac{kg_{water}}{kg_{air}}\frac{m}{s}\right]$ → q
momentum	F	$= \frac{\tilde{F}}{\rho_{air}}$	$\left[\frac{m}{s}\frac{m}{s}\right]$
pollutant	χ	$= \frac{\tilde{\chi}}{\rho_{air}}$	$\left[\frac{kg_{pollutant}}{kg_{air}}\frac{m}{s}\right]$

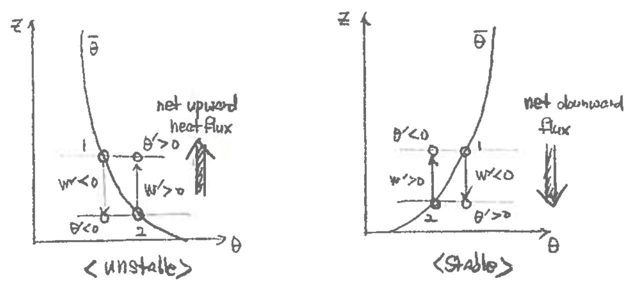
- wind speed → mass & momentum flux
- temp & wind speed → heat flux
- Specific humidity (q) & wind speed → moisture flux
- concentration and wind speed or mass ratio (ppm) and wind speed → pollutant flux

⇒ fluxes are vectors
→ momentum flux is a tensor.

- The flux associated with the mean wind (i.e advection)

vertical kinematic advective heat flux $= \overline{W}\cdot\overline{\theta}$
vertical " " moisture flux $= \overline{W}\cdot\overline{q}$
x-direction kinematic advective heat flux $= \overline{U}\cdot\overline{\theta}$
vertical kinematic advective flux of U-momentum $= \overline{W}\cdot\overline{U}$

2.7 Eddy flux ↔ advective flux.



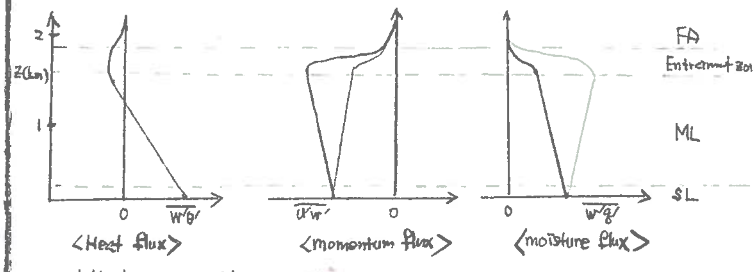
- various kinds of eddy flux

vertical kinematic eddy heat flux $= \overline{w'\theta'}$
" " moisture flux $= \overline{w'q'}$
x-direction kinematic eddy heat flux $= \overline{u'\theta'}$
vertical kinematic eddy flux of U-momentum $= \overline{u'w'}$

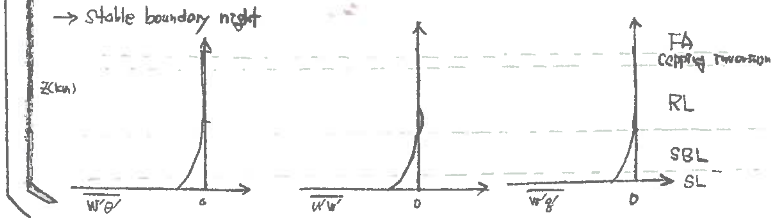
※ below we drop the "kinematic".

- Idealized vertical soundings of turbulent fluxes

→ Convective mixed layer



→ Stable boundary night



2.8 Summation notation (Einstein's summation notation)

m, n, q : integer variable indices (1, 2 or 3)
A_m : a generic velocity vector → $A_1 = U$, $A_2 = V$, $A_3 = W$
X_m : a generic component of distance. → $X_1 = x$, $X_2 = y$, $X_3 = z$
δ_m : a generic unit vector → $\delta_1 = \vec{i}$, $\delta_2 = \vec{j}$, $\delta_3 = \vec{k}$

- A variable with : no free (unsummed) indices = scalar
1 free index = vector
2 free indices = tensor

- Kronecker delta (scalar)

$$\delta_{mn} = \begin{cases} 1 & \text{for } m=n \\ 0 & \text{for } m\neq n \end{cases}$$

- Alternating unit tensor (scalar)

$$\varepsilon_{mnq} = \begin{cases} 1 & \text{for } mnq = 123,\ 231 \text{ or } 312 \\ -1 & \text{for } mnq = 321,\ 213 \text{ or } 132 \\ 0 & \text{for any two or more indices alike} \end{cases}$$

- Three rules

(a) Whenever two identical indices appear in the same one term, it is implied that there is a sum of that term over each value (1, 2 and 3) of the repeated index
(b) Whenever one index appears unsummed (free) in a term, then that same index must appear unsummed in all terms in that eq. Hence, that eq effectively represents 3 eqs for each value of the unsummed index.
(c) The same index cannot appear more then twice in one term

- Ex : eq of motion

$$\frac{\partial A_m}{\partial t} + B_n\frac{\partial A_m}{\partial X_n} = -\delta_{m3}g + f_c\varepsilon_{mn3}B_n - \frac{1}{\rho}\frac{\partial P}{\partial X_m} + \frac{1}{\rho}\left[\frac{\partial \tau_{mn}}{\partial X_n}\right]$$

Sol)
$$\frac{\partial A_m}{\partial t} + B_1\frac{\partial A_m}{\partial X_1} + B_2\frac{\partial A_m}{\partial X_2} + B_3\frac{\partial A_m}{\partial X_3} = -\delta_{m3}g + f_c\varepsilon_{m13}B_1 + f_c\varepsilon_{m23}B_2 + \cancel{f_c\varepsilon_{m33}}B_3 - \frac{1}{\rho}\frac{\partial P}{\partial X_m} + \frac{1}{\rho}\left[\frac{\partial \tau_{m1}}{\partial X_1} + \frac{\partial \tau_{m2}}{\partial X_2} + \frac{\partial \tau_{m3}}{\partial X_3}\right]$$

For m=1
$$\frac{\partial A_1}{\partial t} + B_1\frac{\partial A_1}{\partial X_1} + B_2\frac{\partial A_1}{\partial X_2} + B_3\frac{\partial A_1}{\partial X_3} = \cancel{-\delta_{13}g} + \cancel{f_c\varepsilon_{113}}B_1 + f_c\underset{1}{\boxed{\varepsilon_{123}}}B_2 - \frac{1}{\rho}\frac{\partial P}{\partial X_1} + \frac{1}{\rho}\left[\frac{\partial \tau_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + \frac{\partial \tau_{13}}{\partial X_3}\right]$$

For m=2
$$\frac{\partial A_2}{\partial t} + B_1\frac{\partial A_2}{\partial X_1} + B_2\frac{\partial A_2}{\partial X_2} + B_3\frac{\partial A_2}{\partial X_3} = f_c\overset{-1}{\boxed{\varepsilon_{213}}}B_1 - \frac{1}{\rho}\frac{\partial P}{\partial X_2} + \frac{1}{\rho}\left[\frac{\partial \tau_{21}}{\partial X_1} + \frac{\partial \tau_{22}}{\partial X_2} + \frac{\partial \tau_{23}}{\partial X_3}\right]$$

For m=3
$$\frac{\partial A_3}{\partial t} + B_1\frac{\partial A_3}{\partial X_1} + B_2\frac{\partial A_3}{\partial X_2} + B_3\frac{\partial A_3}{\partial X_3} = -\delta_{33}g - \frac{1}{\rho}\frac{\partial P}{\partial X_3} + \frac{1}{\rho}\left[\frac{\partial \tau_{31}}{\partial X_1} + \frac{\partial \tau_{32}}{\partial X_2} + \frac{\partial \tau_{33}}{\partial X_3}\right]$$

therefore,

$$\begin{cases} \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z} = +f_c V-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{1}{\rho}\left[\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right] \\ \frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z} = -f_c U-\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{1}{\rho}\left[\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}\right] \\ \frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z} = -g-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{1}{\rho}\left(\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right] \end{cases}$$

☆ $$\Rightarrow \frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j} = -\delta_{i3}g+f_c\varepsilon_{ij3}U_j-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\frac{1}{\rho}\left[\frac{\partial \tau_{ij}}{\partial x_j}\right]$$

- Comparison with vector notation

$$\begin{cases} \underset{\sim}{A} \equiv A_m\delta_m \\ \delta_m\cdot\delta_n \equiv \delta_{mn} \\ \delta_m\times\delta_n \equiv \varepsilon_{mnq}\delta_q \\ \nabla(\ ) \equiv \delta_m\frac{\partial(\ )}{\partial x_m} \end{cases}$$

$$\underset{\sim}{A}\cdot\underset{\sim}{B} = A_m B_m \qquad \nabla\cdot\underset{\sim}{A} = \frac{\partial A_m}{\partial x_m}$$

$$\underset{\sim}{A}\times\underset{\sim}{B} = \varepsilon_{mnq}A_m B_n \delta_q$$

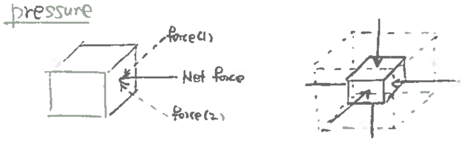
2.9 Stress (Three types : pressure, Reynolds stress, viscous shear stress)

- momentum flux is analogous to a stress.
- Stress : the force tending to produce "deformation" in a body.
  - ↳ force/unit area
- Pressure (a fluid at rest) → scalar
- Reynolds stress → tensor → ex) $-\rho\overline{u'w'}$
  - → exists only when the fluid is in turbulent motion
  - → turbulent momentum flux
- Viscous stress → $\tau_{ij} = \mu\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)+\left(\mu_B-\frac{2}{3}\mu\right)\frac{\partial U_k}{\partial x_k}\delta_{ij}$
  - → exists when there are shearing motions in the fluid

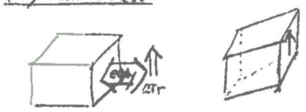
Comparison!

Initial state

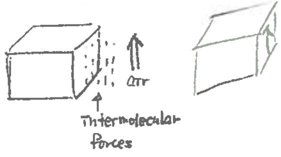
pressure



Reynolds stress



viscous stress



2.10 Friction velocity ← forced-convection scales variables

- During situations where turbulence is generated or modulated by wind shear near the ground, the magnitude of the surface Reynolds' stress proves to be an important scaling variable.

$$\tau_{xz} = -\bar{\rho}\,\overline{u'w'_s} \quad , \quad \tau_{yz} = -\bar{\rho}\,\overline{v'w'_s}$$

$$|\tau_{\text{Reynolds}}| = \left[\tau_{xz}^2+\tau_{yz}^2\right]^{1/2}$$

→ friction velocity (U_*) : velocity scale

✶ $$\boxed{U_*^2 \equiv \left[\overline{u'w'_s}^2+\overline{v'w'_s}^2\right]^{1/2} = |\tau_{\text{Reynolds}}|/\bar{\rho}}$$ ✓

cf) surface layer temperature scale $\theta_*^{SL} = \frac{-\overline{w'\theta'_s}}{U_*}$

" " humidity scale $q_*^{SL} = \frac{-\overline{w'q'_s}}{U_*}$

Ch3 Application of the governing eqs to turbulent flow

3.1 Methodology

Step1. Identify the basic governing eqs that apply to the BL

Step 2. Expand the total derivatives into local and advective contributions

)→ Ch3.2

Step3. Expand dependent variables within those eqs into mean and turbulent (perturbation) parts

Step4. Apply Reynolds averaging to get the eqs for mean variables within a turbulent flow

Step5. Add the continuity eq. to put the result into flux form

→ Ch3.4

Step6. Subtract the eqs of step 5 from the corresponding ones of step3 to get eqs for the turbulent departures from the mean

← Ch4

Step7. Multiply the results of step 6 by other turbulent quantities and Reynolds average to yield prognostic eqs for turbulence statistics such as kinematic flux or TKE.

← ch5

3.2 Basic governing eqs.

(1) Eq of state (ideal gas law) ✓

$$p = \rho_{air} R T_v$$

(2) Conservation of mass (continuity eq.)

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho U_j)}{\partial x_j} = 0 \quad \text{or} \quad \frac{d\rho}{dt}+\rho\frac{\partial U_j}{\partial x_j} = 0$$ ✓

→ incompressibility approximation

$$\frac{\partial U_j}{\partial x_j} = 0$$

(3) Conservation of momentum (Newton's Second law)

$$\underset{\text{I}}{\frac{\partial U_i}{\partial t}}+\underset{\text{II}}{U_j\frac{\partial U_i}{\partial x_j}} = \underset{\text{III}}{-\delta_{i3}g}\ \underset{\text{IV}}{-2\varepsilon_{ijk}\Omega_j U_k}\ \underset{\text{V}}{-\frac{1}{\rho}\frac{\partial P}{\partial x_i}}+\underset{\text{VI}}{\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}}$$

Term I : storage of momentum (inertia)
Term II : advection
Term III : gravity. act vertically
Term IV : Coriolis effects
Term V : PGF
Term VI : influence of viscous stress

→ Term VI : To a close approx

$$\text{Term VI} = \left(\frac{1}{\rho}\right)\frac{\partial}{\partial x_j}\left\{\mu\left[\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right]-\frac{2}{3}\mu\left[\frac{\partial U_k}{\partial x_k}\right]\delta_{ij}\right\}$$

$$= \left(\frac{\mu}{\rho}\right)\left\{\frac{\partial^2 U_i}{\partial x_j^2}+\frac{\partial}{\partial x_i}\left[\frac{\partial U_j}{\partial x_j}\right]-\frac{2}{3}\frac{\partial}{\partial x_i}\left[\frac{\partial U_k}{\partial x_k}\right]\right\}$$

assuming incompressibility

$$\text{Term VI} = \nu\frac{\partial^2 U_i}{\partial x_j^2}$$

kinematic viscosity

permutation tensor

$$\therefore \frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j} = -\delta_{i3}g+f_c\varepsilon_{ij3}U_j-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 U_i}{\partial x_j^2}$$ ✓

(4) Conservation of moisture

q_T : total specific humidity.

assuming incompressibility

$$\underset{\text{I}}{\frac{\partial q_T}{\partial t}}+\underset{\text{II}}{U_j\frac{\partial q_T}{\partial x_j}} = \underset{\text{VI}}{\nu_q\frac{\partial^2 q_T}{\partial x_j^2}}+\underset{\text{VII}}{\frac{S_{q_T}}{\rho_{air}}}$$

ν_q → molecular diffusivity for water vapor in the air

S_{q_T} → net moisture source term (sources − sinks

$q_T = q$ (specific humidity, vapor) + q_L (liquid water content, non-vapor)

$$S_{q_T} = S_q + S_{q_L}$$

$$\begin{cases} \frac{\partial q}{\partial t}+U_j\frac{\partial q}{\partial x_j} = \nu_q\frac{\partial^2 q}{\partial x_j^2}+\frac{S_q}{\rho_{air}}+\frac{E}{\rho_{air}} \\ \frac{\partial q_L}{\partial t}+U_j\frac{\partial q}{\partial x_j} = \qquad +\frac{S_{q_L}}{\rho_{air}}-\frac{E}{\rho_{air}} \end{cases}$$

I II VI VII VIII

E → the mass of water vapor per unit volume per unit time being created by a phase change from liquid or sol

Term VII : a net body source term
Term VIII : the conversion of solid or liquid into vapor

(5) Conservation of heat (first law of thermodynamics)

→ Conservation of enthalpy.

$$\underbrace{\frac{\partial\theta}{\partial t}}_{I}+\underbrace{U_j\frac{\partial\theta}{\partial x_j}}_{II}=\underbrace{\nu_\theta\frac{\partial^2\theta}{\partial x_j^2}}_{VI}-\underbrace{\frac{1}{\rho C_p}\left(\frac{\partial Q_j^*}{\partial x_j}\right)}_{VII}-\underbrace{\frac{L_pE}{\rho C_p}}_{VIII}$$

ν_θ: thermal diffusivity

Q_j^*: component of net radiation in the jth direction

L_p: latent heat associated with the phase change of E

VII: main diabatic source/sink term

(6) Conservation of a scalar quantity

C : concentration (mass per volumn)

$$\underbrace{\frac{\partial C}{\partial t}}_{I}+\underbrace{U_j\frac{\partial C}{\partial x_j}}_{II}=\underbrace{\nu_c\frac{\partial^2 C}{\partial x_j^2}}_{VI}+\underbrace{S_c}_{VII}$$

ν_c: molecular diffusivity of constituent C

S_c: body source term

3.3 Simplications, Approximations and Scaling arguments.

Shallow motion approximation

1) the vertical depth scale of density variations in the BL is much shallower than the scale depth of the lower atmosphere.

2) advection and divergence of mass at a fixed point approximately balance, leaving only slow or zero variations of density with time.

3) the perturbation magnitudes of ρ, T and p are much less than their respective mean values.

Shallow convection approximation

requires all of the conditions above

4) the mean lapse rate ($\partial T/\partial z$) can be negative, zero, or even slightly positive. For the statically stable positive case, $(\partial T/\partial z) \ll g/R\ (\approx 0.0345\,K/m)$

5) the magnitude of the vertical perturbation pressure gradient term must be of the same order or less than the magnitude of the buoyancy term in the eq. of motion.

(1) eq. of state

$\rho=\bar\rho+\rho'$, $T_v=\overline{T_v}+T_v'$, $p=\bar p+p'$

$$\frac{\bar p}{R}+\frac{p'}{R}=(\bar\rho+\rho')(\overline{T_v}+T_v')$$

$$=\bar\rho\overline{T_v}+\rho'\overline{T_v}+\bar\rho T_v'+\rho'T_v' \quad ①$$

→ Reynolds averaging

$$\frac{\bar p}{R}=\bar\rho\overline{T_v}+\underbrace{\overline{\rho'T_v'}}_{\text{small!}} \Rightarrow \frac{\bar p}{R}=\bar\rho\overline{T_v} \quad ②$$

① − ② ⇒

$$\frac{p'}{R}=\rho'\overline{T_v}+\bar\rho T_v'+\rho'T_v' \quad \therefore (\text{divide by } ②)$$

$$\frac{p'}{\bar p}=\frac{\rho'}{\bar\rho}+\frac{T_v'}{\overline{T_v}}+\frac{\rho'T_v'}{\bar\rho\overline{T_v}} \rightarrow \text{small}$$

⇒ linearized perturbation ideal gas law

$$\frac{p'}{\bar p}=\frac{\rho'}{\bar\rho}+\frac{T_v'}{\overline{T_v}}$$

→ using the shallow convection approx. ($\frac{p'}{\bar p}$ small)

$$\frac{\rho'}{\bar\rho}=-\frac{T_v'}{\overline{T_v}}$$

using Poissons relationship

$$\frac{\rho'}{\bar\rho}=-\frac{\theta_v'}{\overline{\theta_v}}$$

(2) flux form of advection terms.

Advection term $= U_j\dfrac{\partial\xi}{\partial x_j}$ ← any variable.

using continuity eq

Advective term $= \dfrac{\partial(\xi U_j)}{\partial x_j}$ ← flux form

(3) Conservation of momentum

⊙ Vertical component

$$\frac{dW}{dt}=-g-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\frac{\partial^2 W}{\partial x_j^2}$$

multiply by ρ. ∵ ($\rho=\bar\rho+\rho'$, $W=\overline{W}+w'$, $p=\bar p+p'$)

$$(\bar\rho+\rho')\frac{d(\overline{W}+w')}{dt}=-(\bar\rho+\rho')g-\frac{\partial(\bar p+p')}{\partial z}+\mu\frac{\partial^2(\overline{W}+w')}{\partial x_j^2}$$

Divide by $\bar\rho$

$$\underbrace{\left(1+\frac{\rho'}{\bar\rho}\right)}_{\approx 1}\frac{d(\overline{W}+w')}{dt}=-\frac{\rho'}{\bar\rho}g-\frac{1}{\bar\rho}\frac{\partial p'}{\partial z}+\nu\frac{\partial^2(\overline{W}+w')}{\partial x_j^2}-\frac{1}{\bar\rho}\underbrace{\left[\frac{\partial\bar p}{\partial z}+\bar\rho g\right]}_{\to 0 \text{ by hydrostatic}}$$

→ Boussinesq approximation

The process of neglecting density variations in the inertia term, but remaining it in the buoyancy (gravity) term

⇒

$$\frac{d(\overline{W}+w')}{dt}=-\frac{\rho'}{\bar\rho}g-\frac{1}{\bar\rho}\frac{\partial p'}{\partial z}+\nu\frac{\partial^2(\overline{W}+w')}{\partial x_j^2}$$

→ A prerequisite for the Boussinesq approximation is that the shallow convection conditions be satisfied.

<Practical Application of the Boussinesq approximation>

$$\rho\rightarrow\bar\rho\ ,\quad g\rightarrow g-\frac{\theta_v'}{\overline{\theta_v}}g$$

→ $\overline{W}\triangleq 0$

$$\therefore \frac{dw'}{dt}=\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g-\frac{1}{\bar\rho}\frac{\partial p'}{\partial z}+\nu\frac{\partial^2 w'}{\partial x_j^2}$$

$-\frac{1}{\bar\rho}\frac{\partial p'}{\partial z}$ → dynamic presure

→ g' = reduced gravity ≈ 0.01g

② Horizontal component.

$$f_cU_g=-\frac{1}{\rho}\frac{\partial P}{\partial y} \quad + \quad f_cV_g=+\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\underbrace{\frac{dU}{dt}}_{\text{inertia}}=-f_c(V_g-V)+\nu\frac{\partial^2 U}{\partial x_j}$$

$$\frac{dV}{dt}=\underbrace{+f_c(U_g-U)}_{\text{geostrophic departure}}+\underbrace{\nu\frac{\partial^2 V}{\partial x_j^2}}_{\text{viscous shear stress}}$$

⊙ Combined momentum eq

$$\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}=-\varepsilon_{ij3}f_c(U_{gj}-U_j)+\delta_{i3}\left[\frac{\theta_v'}{\overline{\theta_v}}g-\frac{1}{\bar\rho}\frac{\partial p'}{\partial z}\right]+\nu\frac{\partial^2U_i}{\partial x_j^2}$$

U_{gj}: $(U_g, V_g, 0)$

↳ we used shallow convection, incompressibility, hydrostatic and Boussinesq approx.s

(4) Horizontal homogeneity

$$\frac{\partial\bar\xi}{\partial x}=0 \quad + \frac{\partial\bar\xi}{\partial y}=0 \rightarrow \text{horizontal homogeneity.}$$

(5) Reorienting and Rotating the Coord. system

3.4 Eqs for mean variables in a turbulent flow

(1) eq of state $\dfrac{\bar P}{R}=\bar\rho\,\overline{T_v}$

(2) Continuity eq

$$\frac{\partial(\overline{U_j}+u_j')}{\partial x_j}=0 \xrightarrow{\text{eq.}} \frac{\partial\overline{U_j}}{\partial x_j}+\frac{\partial u_j'}{\partial x_j}=0$$

$$\therefore \frac{\partial\overline{U_j}}{\partial x_j}=0$$

(cf) continuity eq for turbulent fluctuations.

$$\frac{\partial u_j'}{\partial x_j}=0$$

(3) Conservation of momentum

$$\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}=-\delta_{i3}\left[g-\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g\right]+f_c\varepsilon_{ij3}U_j-\frac{1}{\bar\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2U_i}{\partial x_j^2}$$

$\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g$ ← Boussinesq approx.

$$\frac{\partial(\overline{U_i}+u_i')}{\partial t}+(\overline{U_j}+u_j')\frac{\partial(\overline{U_i}+u_i')}{\partial x_j}=-\delta_{i3}\left[g-\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g\right]+f_c\varepsilon_{ij3}(\overline{U_j}+u_j')$$
$$-\frac{1}{\bar\rho}\frac{\partial(\bar P+p')}{\partial x_i}+\nu\frac{\partial^2(\overline{U_i}+u_i')}{\partial x_j^2}$$

$$\frac{\partial\overline{U_i}}{\partial t}+\frac{\partial u_i'}{\partial t}+\frac{\overline{U_j}\partial\overline{U_i}}{\partial x_j}+\frac{\overline{U_j}\partial u_i'}{\partial x_j}+\frac{u_j'\partial\overline{U_i}}{\partial x_j}+\frac{u_j'\partial u_i'}{\partial x_j}=-\delta_{i3}g+\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g$$
$$+f_c\varepsilon_{ij3}\overline{U_j}+f_c\varepsilon_{ij3}u_j'-\frac{1}{\bar\rho}\frac{\partial\bar P}{\partial x_i}-\frac{1}{\bar\rho}\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2\overline{U_i}}{\partial x_j^2}+\nu\frac{\partial^2u_i'}{\partial x_j^2}$$

→ Reynolds averaging

$$\frac{\partial\overline{U_i}}{\partial t}+\frac{\overline{U_j}\partial\overline{U_i}}{\partial x_j}+\frac{\overline{u_j'\partial u_i'}}{\partial x_j}=-\delta_{i3}g+f_c\varepsilon_{ij3}\overline{U_j}-\frac{1}{\bar\rho}\frac{\partial\bar P}{\partial x_i}+\nu\frac{\partial^2\overline{U_i}}{\partial x_j^2}$$

$$\therefore\left(u_i' \frac{\partial u_j'}{\partial x_j} = 0 \right)$$

$$\therefore \frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{U_i}}{\partial x_j^2} - \frac{\partial \left(\overline{u_i' u_j'}\right)}{\partial x_j}$$

I II III IV V VI X

$$\frac{1}{\overline{\rho}}\frac{\partial \tau_{ij\,Reynolds}}{\partial x_j}$$

$$\tau_{ij\,Reynolds} = -\overline{\rho}\,\overline{u_i' u_j'}$$

- I : Storage of mean momentum (inertia)
- II : advection of mean momentum by the mean wind
- III : gravity vertical only
- IV : Coriolis effect
- V : mean PGF
- VI : viscous stress on the mean motion
- X : Reynold's stress on the mean motion (divergence of turbulent momentum flux)

(4) Conservation of moisture.

$$\frac{\partial \overline{q_T}}{\partial t} + \frac{\partial q_T'}{\partial t} + \overline{U_j}\frac{\partial \overline{q_T}}{\partial x_j} + \overline{U_j}\frac{\partial q_T'}{\partial x_j} + u_j'\frac{\partial \overline{q_T}}{\partial x_j} + u_j'\frac{\partial q_T'}{\partial x_j} = \frac{\nu_q \partial^2 \overline{q}}{\partial x_j^2} + \frac{\nu_q \partial^2 q'}{\partial x_j^2} + \frac{S_{qT}}{\overline{\rho}_{air}}$$

→ Reynolds averaging + using turbulent continuity eq.

$$\frac{\partial \overline{q_T}}{\partial t} + \frac{\overline{U_j}\partial \overline{q_T}}{\partial x_j} = \frac{\nu_q \partial^2 \overline{q}}{\partial x_j} + \frac{S_{qT}}{\overline{\rho}_{air}} - \frac{\partial\left(\overline{u_j' q_T'}\right)}{\partial x_j}$$

I II VI VII X

VI ↳ diffusion

- VII : mean net body source term for additional moisture processes
- X : divergence of turbulent total moisture flux.

(5) Conservation of heat

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\partial \theta'}{\partial t} + \frac{\overline{U_j}\partial \overline{\theta}}{\partial x_j} + \frac{\overline{U_j}\partial \theta'}{\partial x_j} + \frac{u_j'\partial \overline{\theta}}{\partial x_j} + \frac{u_j'\partial \theta'}{\partial x_j} =$$

$$\frac{\nu_\theta \partial^2 \overline{\theta}}{\partial x_j^2} + \frac{\nu_\theta \partial^2 \theta'}{\partial x_j^2} - \frac{1}{\overline{\rho} C_p}\frac{\partial \overline{Q_j^*}}{\partial x_j} - \frac{1}{\overline{\rho} C_p}\frac{\partial Q_j^{*\prime}}{\partial x_j} - \frac{L_v E}{\overline{\rho} C_p}$$

→ Reynolds average + using turbulent continuity eq

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\overline{U_j}\partial \overline{\theta}}{\partial x_j} = \frac{\nu_\theta \partial^2 \overline{\theta}}{\partial x_j^2} - \frac{1}{\overline{\rho} C_p}\frac{\partial \overline{Q_j^*}}{\partial x_j} - \frac{L_v E}{\overline{\rho} C_p} - \frac{\partial\left(\overline{u_j'\theta'}\right)}{\partial x_j}$$

I II VI VII VIII X

VI ↳ conduction

- VIII : body source term associated with latent heat release.
- X : divergence of turbulent heat flux.

(6) Conservation of a scalar quantity (same as above)

$$\frac{\partial \overline{C}}{\partial t} + \frac{\overline{U_j}\partial \overline{C}}{\partial x_j} = \frac{\nu_c \partial^2 \overline{C}}{\partial x_j^2} + S_c - \frac{\partial\left(\overline{u_j' C'}\right)}{\partial x_j}$$

I II VI VII X

VI: diffusion

X ↳ divergence of turbulent tracer flux.

3.5 Summary of eqs, with simplications.

(1) Reynolds number

$$Re \equiv VL/\nu = \rho VL/\mu$$ ⟶ large in BL

where V and L are velocity and length scales in the BL.

→ the ratio of inertial to viscous forcings

(2) Neglect of viscosity for mean motions

- Obs. → molecular diffusion terms are several order of magnitudes smaller

(3) Summary of eqs for mean variables in turbulent flow

→ neglecting molecular diffusion and viscosity, and hydrostatic + Boussinesq approx

$$\frac{\overline{P}}{R} = \overline{\rho}\,\overline{T_v}$$

$$\frac{\partial \overline{U_j}}{\partial x_j} = 0$$

$$\frac{\partial \overline{U}}{\partial t} + \overline{U_j}\frac{\partial \overline{U}}{\partial x_j} = -f_c(\overline{V_g} - \overline{V}) - \frac{\partial\left(\overline{u_j' u'}\right)}{\partial x_j}$$

$$\frac{\partial \overline{V}}{\partial t} + \overline{U_j}\frac{\partial \overline{V}}{\partial x_j} = +f_c(\overline{U_g} - \overline{U}) - \frac{\partial\left(\overline{u_j' v'}\right)}{\partial x_j}$$

$$\frac{\partial \overline{q_T}}{\partial t} + \overline{U_j}\frac{\partial \overline{q_T}}{\partial x_j} = \frac{S_{qT}}{\overline{\rho}_{air}} - \frac{\partial\left(\overline{u_j' q_T'}\right)}{\partial x_j}$$

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{U_j}\frac{\partial \overline{\theta}}{\partial x_j} = -\frac{1}{\overline{\rho} C_p}\left[LE + \frac{\partial \overline{Q_j^*}}{\partial x_j}\right] - \frac{\partial\left(\overline{u_j'\theta'}\right)}{\partial x_j}$$

$$\frac{\partial \overline{C}}{\partial t} + \overline{U_j}\frac{\partial \overline{C}}{\partial x_j} = S_c - \frac{\partial\left(\overline{u_j' c'}\right)}{\partial x_j}$$

I II VII X

Storage advection sundry body forces turbulent flux divergence

Examples

✓ problem 1. Suppose that the turbulent heat flux decreases linearly with height according to $\overline{w'\theta'} = a - bz$ where $a = 0.3 (Kms^{-1})$, $b = 3\times10^{-4} (Ks^{-1})$ If the initial θ profil is an arbitrary shape, then what will be the shape of final profile one hour layer? Neglect subsidence, radiation, latent heating, and assume horizontal homogeneity.

Sol) Neglecting subsidence, radiation and latent heating, mean thermodynamic eq →

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\overline{U}\partial \overline{\theta}}{\partial x} + \frac{\overline{V}\partial \overline{\theta}}{\partial y} = -\frac{\partial(\overline{u'\theta'})}{\partial x} - \frac{\partial(\overline{v\theta'})}{\partial y} - \frac{\partial(\overline{w'\theta'})}{\partial z}$$

Assuming horizontal homogeneity →

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial(\overline{w'\theta'})}{\partial z}$$ ⟵ a-bz

$$= +b$$

$$\overline{\theta}|_t = \overline{\theta}|_{t_0} + b(t - t_0)$$

∴ The warming in one hour is $b(t-t_0) = 3\times10^{-4} \times 3600 = 1.08K$

⇒ anytime the heat flux changes linearly with height, the shape of θ profile will be preserved while it warms, regardless of its initial slope

✓ problem 2 If a horizontal wind of 10 m/s is advecting drier air into a region, where horizo moisture gradient is (5 g_water/kg_air)/100km, then what vertical gradient of turbulent moisture flux in the BL is required to maintain a steady-state q? Assume all the water is in the vapor form, and that there is no body source of moistu additional assumptions needed

Sol) Choose the x-axis to be aligned with the mean wind direction for simplicity.

$$\frac{\overline{U}\partial \overline{q}}{\partial x} + \frac{\overline{W}\partial \overline{q}}{\partial z} = -\frac{\partial(\overline{u'q'})}{\partial x} - \frac{\partial(\overline{v'q'})}{\partial y} - \frac{\partial(\overline{w'q'})}{\partial z}$$

Lets assume no subsidence, no horizontal flux gradients

$$\frac{\overline{U}\partial \overline{q}}{\partial x} = -\frac{\partial(\overline{w'q'})}{\partial z}$$

$$10\,m/s \times 5\times10^{-5}\,(g/kg\ m^{-1}) = -\frac{\partial \overline{w'q'}}{\partial z} = -5\times10^{-4}\ g_{water}/kg_{air}\ s^{-1}$$

✱ a decrease of turbulent flux with height (i.e. flux convergence) results in an increase of the mean variable (e.g. T or q) with time

✓ problem 3 Assume a turbulent BL at a latitude of 44°N, where the mean wind is 2 m/s slower than geostrophic (i.e., the wind is subgeostrophic). Neglect Subside and assume horizontal homogeneity and steady state.

a) Find the Reynolds stress divergence necessary to support this velocity deficit

b) If that stress divergence was related to molecular viscosity instead of turbule what curvature in the mean wind profile would be necessary?

Sol) a) Pick a coord system aligned with the stress, for simplicity.

Assuming horizontal homogeneity, steady state, and neglecting subsidence

$$0 = -f_c(\overline{V_g} - \overline{V}) - \frac{\partial \overline{u'w'}}{\partial z}$$

$$-\frac{\partial \overline{u'w'}}{\partial z} = f_c(\overline{V_g} - \overline{V}) = [10^{-4}(s^{-1})][2(m/s)] = 2\times10^{-4}\ ms^{-2}$$

b) viscosity stress $\nu \frac{\partial^2 U}{\partial z^2} = 2\times10^{-4}\ ms^{-2}$

where $\nu = 1.461\times10^{-5}\ m^2s^{-1}$

$$\frac{\partial^2 U}{\partial z^2} = \frac{[2\times10^{-4}(ms^{-2})]}{[1.5\times10^{-5}(m^2s^{-1})]} = 13.33\,(ms)^{-1}$$ ⟶ tremendously large value

✱ Viscous Stress plays a much smaller role than turbulent Reynolds stress in the mean wind eq. However, viscosity is very important for turbulent motions, and can not be neglected.

3.6 Case studies

Ch4. Prognostic eqs for turbulent fluxes and variances

4.0

- In order to use the mean variables (previous chapter) eqs, we can either evaluate the covariances experimentally, or we can derive additional eqs to forecast the covariances
- Variances : information about turbulence energetics and intensities

(Covariances : kinematic turbulent fluxes

4.1 Prognostic eqs for the turbulent departures

(1) Momentum

$$\cancel{\frac{\partial \overline{U_i}}{\partial t}}+\frac{\partial u_i'}{\partial t}+\cancel{\overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j}}+\overline{U_j}\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial \overline{U_i}}{\partial x_j}+u_j'\frac{\partial u_i'}{\partial x_j} = \cancel{-\delta_{i3}g}+\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g$$

$$+\cancel{f_c\varepsilon_{ij3}\overline{U_j}}+f_c\varepsilon_{ij3}u_j'-\cancel{\left(\frac{1}{\overline{\rho}}\right)\frac{\partial \overline{P}}{\partial x_i}}-\left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}+\cancel{\nu\frac{\partial^2\overline{U_i}}{\partial x_j^2}}+\nu\frac{\partial^2 u_i'}{\partial x_j^2}$$

⇒ subtract mean part !

⇒ a prognostic eq. for just the turbulent gust, u_i'

$$\frac{\partial u_i'}{\partial t}+\overline{U_j}\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial \overline{U_i}}{\partial x_j}+u_j'\frac{\partial u_i'}{\partial x_j} = +\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g+f_c\varepsilon_{ij3}u_j'$$

$$-\left(\frac{1}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 u_i'}{\partial x_j^2}+\frac{\partial\overline{(u_i'u_j')}}{\partial x_j} \quad \text{———} \quad ⓐ$$

(2) Moisture (only vapor)

$$\cancel{\frac{\partial \overline{q}}{\partial t}}+\frac{\partial q'}{\partial t}+\cancel{\overline{U_j}\frac{\partial \overline{q}}{\partial x_j}}+\overline{U_j}\frac{\partial q'}{\partial x_j}+u_j'\frac{\partial \overline{q}}{\partial x_j}+u_j'\frac{\partial q'}{\partial x_j}=\cancel{\nu_q\frac{\partial^2\overline{q}}{\partial x_j^2}}+\nu_q\frac{\partial^2 q'}{\partial x_j^2}+\cancel{\frac{S_q+E}{\overline{\rho}_{air}}}$$

⇒ subtract mean !

<assume mean terms only>

$$\frac{\partial q'}{\partial t}+\overline{U_j}\frac{\partial q'}{\partial x_j}+u_j'\frac{\partial \overline{q}}{\partial x_j}+u_j'\frac{\partial q'}{\partial x_j}=\nu_q\frac{\partial^2 q'}{\partial x_j^2}+\frac{\partial\overline{(u_j'q')}}{\partial x_j} \quad \text{———} \quad ⓑ$$

(3) Heat

$$\frac{\partial\theta'}{\partial t}+\overline{U_j}\frac{\partial\theta'}{\partial x_j}+u_j'\frac{\partial\overline{\theta}}{\partial x_j}+u_j'\frac{\partial\theta'}{\partial x_j}=\nu_\theta\frac{\partial^2\theta'}{\partial x_j^2}+\frac{\partial\overline{(u_j'\theta')}}{\partial x_j}-\frac{1}{\overline{\rho}C_p}\frac{\partial Q_j^{*\prime}}{\partial x_j} \quad \text{———} \quad ⓒ$$

(4) A scalar quantity

$$\frac{\partial c'}{\partial t}+\overline{U_j}\frac{\partial c'}{\partial x_j}+u_j'\frac{\partial\overline{C}}{\partial x_j}+u_j'\frac{\partial c'}{\partial x_j}=\nu_c\frac{\partial^2 c'}{\partial x_j^2}+\frac{\partial\overline{(u_j'c')}}{\partial x_j} \quad \text{———} \quad ⓓ$$

4.2 Free convection scaling variables

- learn how to experimental data is scaled for presentation.
- free convection : buoyant convective processes dominate.
  forced convection : mechanical processes dominate,
- For the free-convection case, strong solar heating at the surface creates a pronounced diurnal cycle in turbulence and ML depth. Profiles of heat and moisture flux can be made nondimensional to remove these diurnal changes (normalized with respect to the top of the mixed layer (z_i) and with respect to surface values of the fluxes)
- Length scale : thermal size scales to z_i
- Velocity scale :

  buoyancy flux $\equiv \frac{g}{\overline{\theta_v}}\overline{w'\theta_v'}$

  free convection scaling velocity (= convective velocity scale)

  $$w_* = \left[\frac{g z_i}{\overline{\theta_v}}\left(\overline{w'\theta_v'}\right)_s\right]^{1/3}$$

- Time scale

  free convection time scale

  $$t_* = \frac{z_i}{w_*}$$

- Temperature scale

  temperature scale for the mixed layer, θ_*^{ML}

  $$\theta_*^{ML} = \frac{\left(\overline{w'\theta'}\right)_s}{w_*}$$

- Humidity scale

  mixed layer humidity scale

  $$q_*^{ML} = \frac{\left(\overline{w'q'}\right)_s}{w_*}$$

4.3 Prognostic eqs for variances

(1) Momentum variance

- Basic derivation

$2u_i' * ⓐ \rightarrow$

$$2u_i'\frac{\partial u_i'}{\partial t}+2\overline{U_j}u_i'\frac{\partial u_i'}{\partial x_j}+2u_i'u_j'\frac{\partial \overline{U_i}}{\partial x_j}+2u_i'u_j'\frac{\partial u_i'}{\partial x_j} = +2\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g$$

$$+2f_c\varepsilon_{ij3}u_i'u_j'-2\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}+2\nu u_i'\frac{\partial^2 u_i'}{\partial x_j^2}+2u_i'\frac{\partial\overline{(u_i'u_j')}}{\partial x_j}$$

$$\frac{\partial u_i'^2}{\partial t}+\overline{U_j}\frac{\partial u_i'^2}{\partial x_j}+2u_i'u_j'\frac{\partial \overline{U_i}}{\partial x_j}+u_j'\frac{\partial u_i'^2}{\partial x_j} = \quad \text{same}$$

→ Reynolds averaging

$$\overline{\frac{\partial u_i'^2}{\partial t}}+\overline{\overline{U_j}\frac{\partial u_i'^2}{\partial x_j}}+\overline{2u_i'u_j'\frac{\partial \overline{U_i}}{\partial x_j}}+\boxed{\overline{u_j'\frac{\partial u_i'^2}{\partial x_j}}} = \overline{2\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g}$$

$$+\overline{2f_c\varepsilon_{ij3}u_i'u_j'}-\overline{2\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}}+\overline{2\nu u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}+\cancelto{0}{\overline{2u_i'\frac{\partial\overline{(u_i'u_j')}}{\partial x_j}}}$$

∵ turbulent continuity eq multiplied by $u_i'^2$ and Reynolds averaging.

$$\left(\overline{u_i'^2\frac{\partial u_j'}{\partial x_j}} = 0\right)$$

$$\frac{\partial\overline{u_i'^2}}{\partial t}+\overline{U_j}\frac{\partial\overline{u_i'^2}}{\partial x_j}+2\overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j}+\frac{\partial\overline{(u_j'u_i'^2)}}{\partial x_j} =$$

$$+2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g}+\cancelto{0}{2f_c\varepsilon_{ij3}\overline{u_i'u_j'}}-2\overline{\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}}+2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

⇒ general form of the prognostic eq. for the variance of wind speed.

→ need more simplification

- dissipation

$$\frac{\partial^2\overline{u_i'^2}}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left[\frac{\partial\overline{(u_i'^2)}}{\partial x_j}\right] = \frac{\partial}{\partial x_j}\left[2\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right] = 2\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}+2\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

$$= 2\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}+2\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

$$\Rightarrow 2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} = \cancel{\nu\frac{\partial^2\overline{(u_i'^2)}}{\partial x_j^2}}-2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$

small (scale analysis)

$$\therefore 2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} \cong -2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} = -2\varepsilon$$

where ε : viscous dissipation

$$\varepsilon \equiv +\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$

- pressure perturbations

$$-2\overline{\left(\frac{u_i'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} = -\left(\frac{2}{\overline{\rho}}\right)\frac{\partial\overline{(u_i'p')}}{\partial x_i}+\boxed{\cancelto{0}{2\overline{\left(\frac{p'}{\overline{\rho}}\right)\left[\frac{\partial u_i'}{\partial x_i}\right]}}}$$

→ for any one component, it is nonzero…

↳ pressure redistribution term

or return-to-isotropy term

Smaller size eddies are more isotropic than larger ones

- Coriolis term (= 0)

$$2f_c\varepsilon_{ij3}\overline{u_i'u_j'} = 2f_c\underset{-1}{\varepsilon_{213}}\overline{u_2'u_1'}+2f_c\underset{1}{\varepsilon_{123}}\overline{u_1'u_2'} = 0$$

⇒ Coriolis force can not generate TKE

- Simplified velocity variance budget eq.

$$\boxed{\frac{\partial\overline{u_i'^2}}{\partial t}+\overline{U_j}\frac{\partial\overline{u_i'^2}}{\partial x_j} = +2\delta_{i3}\frac{g\,\overline{u_i'\theta_v'}}{\overline{\theta_v}}-2\overline{u_i'u_j'}\frac{\partial\overline{U_i}}{\partial x_j}-\frac{\partial\overline{(u_j'u_i'^2)}}{\partial x_j}-\frac{2}{\overline{\rho}}\frac{\partial\overline{(u_i'p')}}{\partial x_i}-2\varepsilon}$$

I II III IV V VI VII

I : local storage of variance

II : advection of variance by the mean wind

III : production or loss term, depending on buoyancy flux

IV : production term (momentum flux $\overline{u_i'u_j'}$ is usually negative in BL)

V : turbulent transport term

VI : how variance is redistributed by pressure perturbations (buoyancy or gravity wav

VII : viscous dissipation of velocity variance

⇒ Component eqs.

$$\frac{\partial \overline{u'^2}}{\partial t}+\overline{U_j}\frac{\partial \overline{u'^2}}{\partial x_j} = -2\overline{u'u_j'}\frac{\partial \overline{U}}{\partial x_j}-\frac{\partial(\overline{u_j'u'^2})}{\partial x_j}-\frac{2}{\bar\rho}\frac{\partial(\overline{u'p'})}{\partial x}+\overline{\frac{2p'}{\bar\rho}\frac{\partial u'}{\partial x}}-2\nu\overline{\left(\frac{\partial u'}{\partial x_j}\right)^2}$$

$$\frac{\partial \overline{v'^2}}{\partial t}+\overline{U_j}\frac{\partial \overline{v'^2}}{\partial x_j} = -2\overline{v'u_j'}\frac{\partial \overline{V}}{\partial x_j}-\frac{\partial(\overline{u_j'v'^2})}{\partial x_j}-\frac{2}{\bar\rho}\frac{\partial(\overline{v'p'})}{\partial y}+\overline{\frac{2p'}{\bar\rho}\frac{\partial v'}{\partial y}}-2\nu\overline{\left(\frac{\partial v'}{\partial x_j}\right)^2}$$

$$\frac{\partial \overline{w'^2}}{\partial t}+\overline{U_j}\frac{\partial \overline{w'^2}}{\partial x_j} = \frac{2g(\overline{w'\theta_v'})}{\overline{\theta_v}}-2\overline{w'u_j'}\frac{\partial \overline{W}}{\partial x_j}-\frac{\partial(\overline{u_j'w'^2})}{\partial x_j}-\frac{2}{\bar\rho}\frac{\partial(\overline{w'p'})}{\partial z}+\overline{\frac{2p'}{\bar\rho}\frac{\partial w'}{\partial z}}-2\nu\overline{\left(\frac{\partial w'}{\partial x_j}\right)^2}$$

I II III IV V VI VIII VII

VIII : pressure redistribution (return-to-isotropy term)

(2) Moisture variance

- Budget equation

$2q'$ * moisture eq ⇒ (perturbation)

$$\frac{\partial q'^2}{\partial t}+\overline{U_j}\frac{\partial q'^2}{\partial x_j}+2q'u_j'\frac{\partial \bar q}{\partial x_j}+u_j'\frac{\partial q'^2}{\partial x_j}=2q'\nu_q\frac{\partial^2 q'}{\partial x_j^2}+2q'\frac{\partial(\overline{u_j'q'})}{\partial x_j}$$

→ Reynolds average

$$\frac{\partial \overline{q'^2}}{\partial t}+\overline{U_j}\frac{\partial \overline{q'^2}}{\partial x_j}+2\overline{q'u_j'}\frac{\partial \bar q}{\partial x_j}+\overline{u_j'\frac{\partial q'^2}{\partial x_j}}=\overline{2q'\nu_q\frac{\partial^2 q'}{\partial x_j^2}}$$

$\because\left(\overline{q'^2\frac{\partial u_j'}{\partial x_j}}=0\right.$ ← averaged turbulent continuity eq multiplied by q'^2)

$$\frac{\partial \overline{q'^2}}{\partial t}+\overline{U_j}\frac{\partial \overline{q'^2}}{\partial x_j}=-2\overline{q'u_j'}\frac{\partial \bar q}{\partial x_j}-\frac{\partial(\overline{u_j'q'^2})}{\partial x_j}+2\nu_q\overline{q'\frac{\partial^2 q'}{\partial x_j^2}}$$

(Similar as before $\varepsilon_q=\nu_q\overline{\left(\frac{\partial q'}{\partial x_j}\right)^2}$)

∴ the prognostic equation for specific humidity variance

$$\boxed{\frac{\partial \overline{q'^2}}{\partial t}+\overline{U_j}\frac{\partial \overline{q'^2}}{\partial x_j}=-2\overline{q'u_j'}\frac{\partial \bar q}{\partial x_j}-\frac{\partial(\overline{u_j'q'^2})}{\partial x_j}-2\varepsilon_q}$$ ✓

I II IV V VII

I : local storage of humidity variance

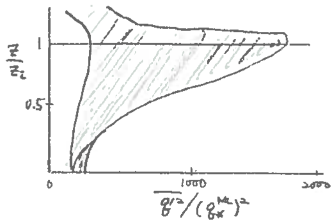
II : advection of humidity variance by the mean wind.

IV : production term (turbulent motions occurring within a mean moisture gradient)

V : turbulent transport of humidity variance

VII : molecular dissipation

- modeled vertical profiles of dimensionless specific-humidity variance



(3) Heat (potential temperature variance)

- Budget eqs

Same as before, $2\theta'$ * turbulent heat eq. and use the product rule of calculus, Reynolds average, put into flux form, neglect molecular diffusion but remain the molecular dissipation ⟶

$$\boxed{\frac{\partial \overline{\theta'^2}}{\partial t}+\overline{U_j}\frac{\partial \overline{\theta'^2}}{\partial x_j}=-2\overline{\theta'u_j'}\frac{\partial \bar\theta}{\partial x_j}-\frac{\partial(\overline{u_j'\theta'^2})}{\partial x_j}-2\varepsilon_\theta-\left(\frac{2}{\bar\rho C_p}\right)\overline{\theta'\frac{\partial Q_j^{*\prime}}{\partial x_j}}}$$ ✓

I II IV V VII VIII

VIII : the radiation destruction term (ε_R)

(4) A scalar quantity (Tracer concentration variance)

$2C'$ * eq ⇒ (same as above)

$$\boxed{\frac{\partial \overline{C'^2}}{\partial t}+\overline{U_j}\frac{\partial \overline{C'^2}}{\partial x_j}=-2\overline{C'u_j'}\frac{\partial \bar C}{\partial x_j}-\frac{\partial(\overline{u_j'C'^2})}{\partial x_j}-2\varepsilon_c}$$ ✓

4.4 Prognostic eqs for turbulent fluxes.

(1) Momentum flux

- Budget eq. : Two perturbation eqs are combined to produce flux eqs.

u_k' * perturbation momentum eq + Reynolds average →

$$\overline{u_k'\frac{\partial u_i'}{\partial t}}+\overline{u_k'\overline{U_j}\frac{\partial u_i'}{\partial x_j}}+\overline{u_k'u_j'\frac{\partial \overline{U_i}}{\partial x_j}}+\overline{u_k'u_j'\frac{\partial u_i'}{\partial x_j}}=$$

$$+\delta_{i3}\overline{u_k'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g+f_c\varepsilon_{ij3}\overline{u_k'u_j'}-\overline{\left(\frac{u_k'}{\bar\rho}\right)\frac{\partial p'}{\partial x_i}}+\nu\overline{u_k'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

For the second eq, interchange the i and k indices.

$$\overline{u_i'\frac{\partial u_k'}{\partial t}}+\overline{u_i'\overline{U_j}\frac{\partial u_k'}{\partial x_j}}+\overline{u_i'u_j'\frac{\partial \overline{U_k}}{\partial x_j}}+\overline{u_i'u_j'\frac{\partial u_k'}{\partial x_j}}=$$

$$+\delta_{k3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}+f_c\varepsilon_{kj3}\overline{u_i'u_j'}-\overline{\left(\frac{u_i'}{\bar\rho}\right)\frac{\partial p'}{\partial x_k}}+\nu\overline{u_i'\frac{\partial^2 u_k'}{\partial x_j^2}}$$

→ Add these two eqs together. + use the product rule of calculus

$$\frac{\partial \overline{u_i'u_k'}}{\partial t}+\frac{\overline{U_j}\,\partial \overline{u_i'u_k'}}{\partial x_j}+\overline{u_i'u_j'}\frac{\partial \overline{U_k}}{\partial x_j}+\overline{u_k'u_j'}\frac{\partial \overline{U_i}}{\partial x_j}+\underbrace{\frac{\overline{u_j'\,\partial u_i'u_k'}}{\partial x_j}}_{\text{flux form}}$$

$$=\delta_{k3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g+\delta_{i3}\overline{u_k'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g+f_c\varepsilon_{kj3}\overline{u_i'u_j'}+f_c\varepsilon_{ij3}\overline{u_k'u_j'}$$

$$\underbrace{-\overline{\left(\frac{u_i'}{\bar\rho}\right)\frac{\partial p'}{\partial x_k}}-\overline{\left(\frac{u_k'}{\bar\rho}\right)\frac{\partial p'}{\partial x_i}}+\nu\overline{u_i'\frac{\partial^2 u_k'}{\partial x_j^2}}+\nu\overline{u_k'\frac{\partial^2 u_i'}{\partial x_j^2}}}_{\text{different f}}$$

$\because\left(\text{turbulent continuity eq} \longrightarrow \overline{u_i'u_k'\frac{\partial u_j'}{\partial x_j}}=0\right)$

∴ final form for the momentum flux budget equation

✓

$$\frac{\partial(\overline{u_i'u_k'})}{\partial t}+\frac{\overline{U_j}\,\partial(\overline{u_i'u_k'})}{\partial x_j}=-(\overline{u_i'u_j'})\frac{\partial \overline{U_k}}{\partial x_j}-(\overline{u_k'u_j'})\frac{\partial \overline{U_i}}{\partial x_j}-\frac{\partial(\overline{u_i'u_j'u_k'})}{\partial x_j}$$

I II III III IV

$$+\left(\frac{g}{\overline{\theta_v}}\right)[\delta_{k3}\overline{u_i'\theta_v'}+\delta_{i3}\overline{u_k'\theta_v'}]+f_c[\varepsilon_{ki3}\overline{u_i'u_j'}+\varepsilon_{ij3}\overline{u_k'u_j'}]\xrightarrow{}\text{small}$$

V V VI VI

$$-\frac{1}{\bar\rho}\left[\frac{\partial(\overline{p'u_k'})}{\partial x_i}+\frac{\partial(\overline{p'u_i'})}{\partial x_k}\xrightarrow{}\text{small}-\overline{p'\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)}\right]+\frac{\nu\,\partial^2(\overline{u_i'u_k'})}{\partial x_j^2}\xrightarrow{}\text{small}-\frac{2\nu\overline{\partial u_i'\partial u_k'}}{\partial x_j^2}$$

VII VII VIII IX X

I : Storage of momentum flux $\overline{u_i'u_k'}$

II : Advection of momentum flux by the mean wind.

III : Production of momentum flux by the mean wind shears

IV : Transport of momentum flux by turbulent motions (turbulent diffusion)

V : Buoyant production or consumption

VI : Coriolis effects

VII : Transport by the pressure correlation term (pressure diffusion)

VIII : Redistribution by the return-to-isotropy term

IX : molecular diffusion of turbulent momentum flux

X : Viscous dissipation term $=2\varepsilon_{u_iu_k}$

After scaling arguments,

$$\frac{\partial(\overline{u_i'u_k'})}{\partial t}+\frac{\overline{U_j}\,\partial(\overline{u_i'u_k'})}{\partial x_j}=-(\overline{u_i'u_j'})\frac{\partial \overline{U_k}}{\partial x_j}-(\overline{u_k'u_j'})\frac{\partial \overline{U_i}}{\partial x_j}-\frac{\partial(\overline{u_i'u_j'u_k'})}{\partial x_j}$$

$$+\left(\frac{g}{\overline{\theta_v}}\right)[\delta_{k3}\overline{u_i'\theta_v'}+\delta_{i3}\overline{u_k'\theta_v'}]+\overline{\frac{p'}{\bar\rho}\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)}-2\varepsilon_{u_iu_k}$$

↳ 9 separate eqs ⟶ 6 eqs (by symmetries)

(2) Moisture flux

- Budget eqs

For the first eq, start with the momentum perturbation eq. * q' + Reynolds average

$$\overline{q'\frac{\partial u_i'}{\partial t}}+\overline{U_j}\,\overline{q'\frac{\partial u'}{\partial t}}+\overline{q'u_j'}\frac{\partial \overline{U_i}}{\partial x_j}+\overline{q'u_j'\frac{\partial u_i'}{\partial x_j}}=$$

$$+\overline{q'\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g+f_c\varepsilon_{ij3}\overline{u_j'q'}-\overline{\left(\frac{q'}{\bar\rho}\right)\frac{\partial p'}{\partial x_i}}+\nu\overline{q'\frac{\partial^2 u_i'}{\partial x_j^2}}$$

For the second eq, u_i' * moisture perturbation eq & Reynolds average

$$\overline{u_i'\frac{\partial q'}{\partial t}} + \overline{U_j}\,\overline{u_i'\frac{\partial q'}{\partial x_j}} + \overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j} + \overline{u_i'u_j'\frac{\partial q'}{\partial x_j}} = \nu_q \overline{u_i'\frac{\partial^2 q'}{\partial x_j^2}}$$

Add above two eqs, and rearrange (turbulent continuity eq)

$$\frac{\partial(\overline{q'u_i'})}{\partial t} + \overline{U_j}\frac{\partial(\overline{q'u_i'})}{\partial x_j} + \overline{q'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} + \overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\partial(\overline{q'u_j'u_i'})}{\partial x_j} =$$
$$+\delta_{i3}\left(\frac{\overline{q'\theta_v'}}{\overline{\theta_v}}\right)g + f_c\,\varepsilon_{ij3}\,\overline{u_j'q'} - \overline{\left(\frac{q'}{\overline{\rho}}\right)\frac{\partial p'}{\partial x_i}} + \nu\,\overline{q'\frac{\partial^2 u_i'}{\partial x_j^2}} + \nu_q\,\overline{u_i'\frac{\partial^2 q'}{\partial x_j^2}}$$

split the pressure term into two parts, assume $\nu \doteq \nu_q$

$$\therefore \boxed{\begin{aligned}&\frac{\partial(\overline{q'u_i'})}{\partial t} + \overline{U_j}\frac{\partial(\overline{q'u_i'})}{\partial x_j} = -\overline{q'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j} - \frac{\partial(\overline{q'u_j'u_i'})}{\partial x_j}\\&\quad\text{I}\qquad\quad\text{II}\qquad\qquad\qquad\text{III}\qquad\qquad\text{XI}\qquad\qquad\text{IV}\\&+\delta_{i3}\left(\frac{\overline{q'\theta_v'}}{\overline{\theta_v}}\right)g + \cancel{f_c\varepsilon_{ij3}(\overline{u_j'q'})} - \left(\frac{1}{\overline{\rho}}\right)\left[\cancel{\frac{\partial(\overline{p'q'})}{\partial x_i}} - \overline{p'\frac{\partial q'}{\partial x_i}}\right]\\&\qquad\text{V}\qquad\qquad\text{VI}\qquad\qquad\qquad\text{VII}\qquad\quad\text{VIII}\\&+\nu\cancel{\frac{\partial^2(\overline{q'u_i'})}{\partial x_j^2}} - 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial q'}{\partial x_j}\right)} \rightarrow 2\varepsilon_{u_iq}\\&\qquad\text{IX}\qquad\qquad\qquad\text{X}\end{aligned}}$$

XI : a production/loss term related to the mean moisture gradient

* an additional term must be added if the body source is assumed to have perturbations too

⇒ After scaling arguments

$$\frac{\partial(\overline{q'u_i'})}{\partial t} + \overline{U_j}\frac{\partial(\overline{q'u_i'})}{\partial x_j} = -\overline{q'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j} - \frac{\partial(\overline{q'u_j'u_i'})}{\partial x_j}$$
$$+\delta_{i3}\left(\frac{\overline{q'\theta_v'}}{\overline{\theta_v}}\right)g + \frac{1}{\overline{\rho}}\left[\overline{\frac{p'\partial q'}{\partial x_i}}\right] - 2\varepsilon_{u_iq}$$

→ Term V relates the correlation (covariance) between moisture and temperature to the production of moisture flux ✓

(3) Heat flux

· **Budget eqs** : similar to that of the moisture flux

$$\boxed{\begin{aligned}&\frac{\partial(\overline{\theta'u_i'})}{\partial t} + \overline{U_j}\frac{\partial(\overline{\theta'u_i'})}{\partial x_j} = -\overline{\theta'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{\theta}}{\partial x_j} - \frac{\partial(\overline{\theta'u_j'u_i'})}{\partial x_j}\\&\quad\text{I}\qquad\quad\text{II}\qquad\qquad\qquad\text{III}\qquad\qquad\text{XI}\qquad\qquad\text{IV}\\&+\delta_{i3}\left(\frac{\overline{\theta'\theta_v'}}{\overline{\theta_v}}\right)g + \cancel{f_c\varepsilon_{ij3}(\overline{u_j'\theta'})} - \left(\frac{1}{\overline{\rho}}\right)\left[\cancel{\frac{\partial(\overline{p'\theta'})}{\partial x_i}} - \overline{p'\frac{\partial \theta'}{\partial x_i}}\right]\\&\qquad\text{V}\qquad\qquad\text{VI}\qquad\qquad\qquad\text{VII}\qquad\quad\text{VIII}\\&+\cancel{\frac{\nu\partial^2(\overline{\theta'u_i'})}{\partial x_j^2}} - 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial \theta'}{\partial x_j}\right)} - \cancel{\left(\frac{1}{\overline{\rho}C_p}\right)\overline{u_i'\frac{\partial Q_j^{*\prime}}{\partial x_j}}}\\&\qquad\text{IX}\qquad\qquad\quad\text{X}\ \hookrightarrow 2\varepsilon_{u_i\theta}\qquad\qquad\text{XII}\end{aligned}}$$

Term V is often approximated by $\delta_{i3}\,g\left(\frac{\overline{\theta_v'^2}}{\overline{\theta_v}}\right)$

Term XII describes the correlation between velocity fluctuations and with radiation fluctuations

⇒ After scaling

$$\frac{\partial(\overline{\theta'u_i'})}{\partial t} + \overline{U_j}\frac{\partial(\overline{\theta'u_i'})}{\partial x_j} = -\overline{\theta'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{\theta}}{\partial x_j} - \frac{\partial(\overline{\theta'u_j'u_i'})}{\partial x_j}$$
$$+\delta_{i3}\left(\frac{\overline{\theta'\theta_v'}}{\overline{\theta_v}}\right)g + \left(\frac{1}{\overline{\rho}}\right)\left[\overline{\frac{p'\partial\theta'}{\partial x_i}}\right] - 2\varepsilon_{u_i\theta}$$

(4) Flux of a scalar (pollutant or tracer flux)

using the same procedures as before, the pollutant flux budget eq is

$$\boxed{\begin{aligned}&\frac{\partial(\overline{c'u_i'})}{\partial t} + \overline{U_j}\frac{\partial(\overline{c'u_i'})}{\partial x_j} = -\overline{c'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{c}}{\partial x_j} - \frac{\partial(\overline{c'u_j'u_i'})}{\partial x_j}\\&\quad\text{I}\qquad\quad\text{II}\qquad\qquad\qquad\text{III}\qquad\qquad\text{XI}\qquad\qquad\text{IV}\\&+\delta_{i3}\left(\frac{\overline{c'\theta_v'}}{\overline{\theta_v}}\right)g + \cancel{f_c\varepsilon_{ij3}(\overline{u_j'c'})} - \left(\frac{1}{\overline{\rho}}\right)\left[\cancel{\frac{\partial(\overline{p'c'})}{\partial x_i}} - \overline{\frac{p'\partial c'}{\partial x_i}}\right] + \cancel{\frac{\nu\partial^2(\overline{c'u_i'})}{\partial x_j^2}} - 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial c'}{\partial x_j}\right)}\\&\qquad\text{V}\qquad\qquad\text{VI}\qquad\qquad\qquad\text{VII}\qquad\quad\text{VIII}\qquad\qquad\text{IX}\qquad\qquad\text{X}\end{aligned}}$$

⇒ after scaling

$$\frac{\partial(\overline{c'u_i'})}{\partial t} + \overline{U_j}\frac{\partial(\overline{c'u_i'})}{\partial x_j} = -\overline{c'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{c}}{\partial x_j} - \frac{\partial(\overline{c'u_j'u_i'})}{\partial x_j}$$
$$+\delta_{i3}\left(\frac{\overline{c'\theta_v'}}{\overline{\theta_v}}\right)g + \left(\frac{1}{\overline{\rho}}\right)\left[\overline{p'\frac{\partial c'}{\partial x_i}}\right] - 2\varepsilon_{u_ic}$$

(5) Buoyancy flux

buoyancy flux $\equiv \left(\frac{g}{\overline{\theta_v}}\right)(\overline{w'\theta_v'})_s$

→ The flux of virtual potential temperature $\overline{w'\theta_v'}$ is different than the heat flux $\overline{w'\theta'}$.

→ Let's derive a diagnostic eq. for $\overline{w'\theta_v'}$ in terms of $\overline{w'\theta'}$

$$\theta_v \cong \theta[1+0.61r-r_L]$$

$$\overline{\theta_v}+\theta_v' = (\overline{\theta}+\theta')[1+0.61(\overline{r}+r')-(\overline{r_L}+r_L')]$$
$$= \overline{\theta}[1+0.61\overline{r}-\overline{r_L}] + \overline{\theta}[0.61r'-r_L'] + \theta'[1+0.61\overline{r}-\overline{r_L}] + \theta'[0.61r'-r_L']$$

multiplying w' and Reynolds averaging

$$\overline{w'\theta_v'} = \overline{\theta}[0.61(\overline{w'r'})-(\overline{w'r_L'})] + (\overline{w'\theta'})[1+0.61\overline{r}-\overline{r_L}] + \cancel{\overline{w'\theta'[0.61r'-r_L']}} \rightarrow \text{small}$$

* $$\overline{w'\theta_v'} \cong \overline{\theta}[0.61(\overline{w'r'})-(\overline{w'r_L'})] + (\overline{w'\theta'})[1+0.61\overline{r}-\overline{r_L}]$$

For the special case of no liquid water in the air

$$\overline{w'\theta_v'} \cong (\overline{w'\theta'})[1+0.61\overline{r}] + 0.61\overline{\theta}(\overline{w'r'})$$

Ch5. TKE, stability and scaling

5.0

TKE : a measure of the intensity of turbulence.

└→ a starting point for approximations of turbulent diffusion

5.1 The TKE budget derivation

$$\frac{\text{TKE}}{m} = \overline{e} = 0.5(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}) = \frac{1}{2}\overline{u_i'^2}$$

└→ summed velocity variances divided by two

p12 eq. → Scaling parameter

* $$\boxed{\frac{\partial \overline{e}}{\partial t} + \overline{U_j}\frac{\partial \overline{e}}{\partial x_j} = +\delta_{i3}\frac{g}{\overline{\theta_v}}(\overline{u_i'\theta_v'}) - \overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \frac{\partial \overline{u_j'e}}{\partial x_j} - \frac{1}{\overline{\rho}}\frac{\partial(\overline{u_i'p'})}{\partial x_i} - \varepsilon}$$ ——

I II III IV V VI VII

- I : local storage or tendency of TKE
- II : advection of TKE by the mean wind
- III : buoyant production or consumption term ✓
- IV : mechanical or shear production/loss term ✓
- V : turbulent transport of TKE
- VI : pressure correlation term
- VII : viscous dissipation of TKE (conversion of TKE into heat)

If we choose a coordinate system aligned with the mean wind, assume horizontal homogenei and neglect subsidence →

$$\frac{\partial \overline{e}}{\partial t} = \frac{g}{\overline{\theta_v}}(\overline{w'\theta_v'}) - \overline{u'w'}\frac{\partial \overline{U}}{\partial z} - \frac{\partial(\overline{w'e})}{\partial z} - \frac{1}{\overline{\rho}}\frac{\partial(\overline{w'p'})}{\partial z} - \varepsilon \qquad \text{——— ②}$$

Turbulence is dissipative → TKE is not a conserved quantity.

5.2 Contributions to the TKE budget

(1) Term I : Storage.

- The vertical profile of TKE can sometimes increase to a maximum at a height of about $z/z_i \simeq 0.3$ when free convection dominates.
- When strong winds are present, the TKE might be nearly constant with height within the BL, or might decrease slightly with height.
- At night, the TKE often decreases very rapidly with height, from a maximum value just above the surface

(2) Term II : Advection

- Little is known about this term.
- negligible in larger scale
- important in smaller scale

(3) Term III : buoyant production/consumption

- Production
  - → $\overline{w'\theta_v'}$ is positive and decreases roughly linearly with height within the bottom 2/3 of the convective ML
  - → when positive, it represents the effects of thermals in the ML.
  - → sunny days over land or cold air advection over a warmer underlying surface
  - → For cloudy days over land, it can be much smaller
  - → Because this term is so important on days of free convection, it is often used to normalize all the other terms. ☆
  - → Term III = $\boxed{(W_*)^3/z_i}$ at the surface

√ → 5.1 eq ②/$\boxed{(W_*)^3/z_i}$ ⇒ dimensionless form of the TKE budget eq.
(useful for free convection situations)

$$\underset{\text{I}}{\frac{z_i}{W_*^3}\frac{\partial \overline{e}}{\partial t}} = \underset{\text{III}}{\frac{g z_i(\overline{w'\theta_v'})}{W_*^3\,\overline{\theta_v}}} - \underset{\text{IV}}{\frac{z_i\,\overline{u'w'}}{W_*^3}\frac{\partial \overline{U}}{\partial z}} - \underset{\text{V}}{\frac{z_i}{W_*^3}\frac{\partial(\overline{w'e})}{\partial z}} - \underset{\text{VI}}{\frac{z_i}{W_*^3}\frac{1}{\overline{\rho}}\frac{\partial(\overline{w'p'})}{\partial z}} - \underset{\text{VII}}{\frac{z_i\varepsilon}{W_*^3}}$$

- → By definition, the dimensionless Term III is unity at the surface.
- → Eqs that are made dimensionless by dividing by scaling parameters are said to be normalized.
- → only vertical component TKE → "anisotropic"

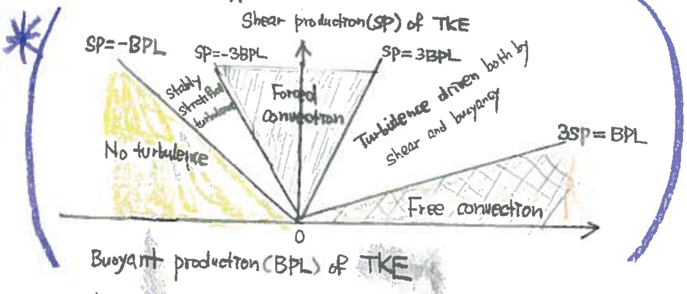
- Consumption
  - → static stability tends to suppress, or consume, TKE.
  - → SBL at night over land

(4) Term IV : Mechanical (Shear) Production.

- When there is a turbulent momentum flux in the presence of mean wind shear, the interaction between the two tends to generate more turbulence. Even though a negative sign precedes Term IV, the momentum flux is usually of opposite sign from the mean shear → production of turbulence
- The greatest wind shear magnitude occurs at the surface. Not surprisingly, the maximum shear production rate also occurs there
- The wind speed frequently varies little with height in the ML above the surface layer, resulting in near zero shear and near zero shear production of turbulence

☆ The relative contributions of the buoyancy and shear terms can be used to classify the nature of convection.

⇒ Free convection scaling is valid when the buoyancy term is much larger than the mechanical term, forced convection scaling is valid when the opposite is true.



- anisotropic → strongest in the horizontal
- Shear generation produces turbulence primarily in the horizontal directions, while buoyant generation produces it primarily in the vertical.

(5) Term V : turbulent transport

- $\overline{w'e}$ represents the vertical turbulent flux of TKE.
- When integrated over the depth of the ML, term V becomes identically zero.
- This term neither creates nor destroys TKE, it just moves or redistributes TKE from one location in the BL to another

(6) Term VI : Pressure Correlation

10/1

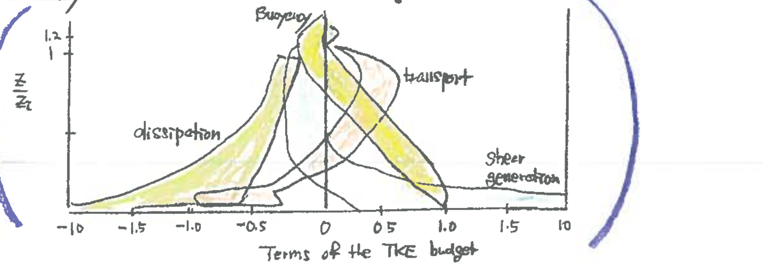
- Turbulence
  - → Static pressure fluctuations are exceedingly difficult to measure in the atmosphere.
  - → residual
- Waves
  - → Work in linear gravity wave theory shows that $\overline{w'p'}$ is equal to the upward flux of wave energy for a vertically propagating internal gravity wave within a statically stable environment. → turbulence energy can be lost from the ML top in the form of internal gravity waves

⇒ the pressure correlation term not only acts to redistribute TKE within the BL but it can also drain energy out of the BL.

(7) Term VII : Dissipation

- molecular destruction of turbulent motions is greatest for the smallest size eddies.
- TKE ↑ → ε↑

*Summary → normalized terms in the TKE eq.



(8) Example

Problem At a height of z = 300m in a 1000m thick mixed layer, the following conditions were observed : $\partial \overline{U}/\partial z = 0.01 s^{-1}$, $\overline{\theta_v} = 25°C$, $\overline{w'\theta_v'} = 0.15 K\,m/s$, $\overline{u'w'} = -0.03 m^2 s^{-2}$. Also, the surface virtual heat flux $(\overline{w'\theta_v'})_s$ is 0.24 Km/s. If the pressure and turbulent transports are neglected, then (a) what dissipation rate is required to maintain a locally steady state at z = 300m ; and (b) what are the values of the normalized TKE terms?

Sol) (a) mean wind → x-axis

P14 eq ② → $\varepsilon = \frac{g}{\overline{\theta_v}}\overline{w'\theta_v'} - \overline{u'w'}\frac{\partial \overline{U}}{\partial z}$

$$= \frac{9.8}{273.15+25}0.15 - (-0.03)(0.01) = 5.23\times10^{-3}\ m^2 s^{-3}$$

(b) $\frac{W_*^3}{z_i} = \frac{g}{\overline{\theta_v}}(\overline{w'\theta_v'})_s = 7.89\times10^{-3} (m^2 s^{-3})$

$$\underset{\text{I}}{0} = \underset{\text{III}}{0.625} + \underset{\text{IV}}{0.038} - \underset{\text{V}}{0} - \underset{\text{VI}}{0} - \underset{\text{VII}}{0.663}$$ ⇒ free convection.

Term I III IV V VI VII

※ local dissipation rate at z = 300m is smaller than the value calculated above

5.3 TKE budget contributions as a function of eddy size → 생략
↳ inertial subrange

5.4 Mean Kinetic energy and its interaction with turbulence

- the production of TKE is accompanied by a corresponding loss of KE from the mean flo
- mean KE per unit mass [$MKE/m = 0.5(\overline{U}^2+\overline{V}^2+\overline{W}^2) = 0.5\overline{U_i}^2$]

ch 3.4.(3) → $\times\overline{U_i}$

$$\underset{\text{I}}{\frac{\partial(0.5\overline{U_i}^2)}{\partial t}} + \underset{\text{II}}{\overline{U_j}\frac{\partial(0.5\overline{U_i}^2)}{\partial x_j}} = \underset{\text{III}}{-g\delta_{i3}\overline{U_i}} + \underset{\text{IV}}{\cancelto{0}{f_c\varepsilon_{ij3}\overline{U_i}\,\overline{U_j}}} - \underset{\text{V}}{\frac{\overline{U_i}}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i}} + \underset{\text{VI}}{\nu\overline{U_i}\frac{\partial^2\overline{U_i}}{\partial x_j^2}} - \underset{\text{X}}{\overline{U_i}\frac{\partial(\overline{u_i'u_j'})}{\partial x_j}}$$

X : the interaction between the mean flow and turbulence

$$\because \left(-\overline{U_i}\frac{\partial\overline{u_i'u_j'}}{\partial x_j} = \overline{u_i'u_j'}\frac{\partial\overline{U_i}}{\partial x_j} - \frac{\partial(\overline{u_i'u_j'}\,\overline{U_i})}{\partial x_j}\right)$$

so

$$\frac{\partial(0.5\overline{U_i}^2)}{\partial t} + \overline{U_j}\frac{\partial(0.5\overline{U_i}^2)}{\partial x_j} = -g\overline{W} - \frac{\overline{U_i}}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_i} + \nu\overline{U_i}\frac{\partial^2\overline{U_i}}{\partial x_j^2} + \overline{u_i'u_j'}\frac{\partial\overline{U_i}}{\partial x_j} - \frac{\partial(\overline{u_i'u_j'}\,\overline{U_i})}{\partial x_j}$$

Compare with p14 eq ①

⇒ The energy that is mechanically produced as turbulence is lost from the mean flow, and vice versa.

5.5 Stability concepts

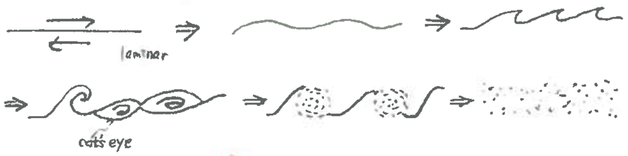
(1) Static stability and convection

- Static stability is a measure of the capability for buoyant convection.
- Local definitions
  → the static stability is determined by the local lapse rate.
  → The local definition frequently fails in convective MLs, because the rise of thermals from near the surface or their descent from cloud top depends on their excess buoyancy and not on the ambient lapse rate.
  → adiabatic lapse rate (in the virtual potential temp. sense)
    ↳ may be statically stable, neutral, or unstable, depending on convection and the buoyancy flux.
    neutral stability → adiabatic lapse rate AND no convection
  → measurement of the local lapse rate alone is insufficient to determine the static stability.
- Nonlocal definitions → examine the stability of the whole layer
  → If $\overline{w'\theta_v'}$ at the surface is positive, or if displaced air parcels will rise from the ground or sink from the cloud top as thermals traveling across a BL, then the whole BL is said to be unstable or convective
  → If $\overline{w'\theta_v'}$ is negative at the surface, or if displaced air parcels return to their starting point, then the BL → stable
  → If, when integrated over the depth of the BL, the mechanical production term in the TKE eq is much larger than the buoyancy term, or if the buoyancy term is near zero, then the BL → neutral (Ekman BL)
  → An alternate determination of static stability is possible if the $\overline{\theta_v}$ profile over the whole BL is known.

(2) Dynamic stability + Kelvin-Helmholtz waves

↳ related with wind

- Schematic diagram of Kelvin-Helmholtz instability



5.6 The Richardson number

(1) Flux Richardson number

- In a statically stable environment
  → buoyant production term of the TKE budget eq is negative, while the mechanical production term is positive.
- The ratio of Term III to Term IV in TKE eq. → flux Richardson # (R_f)

$$R_f = \frac{\left(\frac{g}{\overline{\theta_v}}\right)\overline{w'\theta_v'}}{\overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j}}$$

- Assume horizontal homogeneity and neglect subsidence

$$R_f = \frac{\left(\frac{g}{\overline{\theta_v}}\right)\overline{w'\theta_v'}}{\overline{u'w'}\frac{\partial \overline{U}}{\partial z} + \overline{v'w'}\frac{\partial \overline{V}}{\partial z}}$$

- For statically unstable flows, R_f is usually negative (remember that the denominator is usually negative). For neutral flows, it is zero. For statically stable flows, R_f is positive
- $R_f = +1$ is a critical value
- Flow is turbulent (dynamically unstable) when $R_f < +1$
  Flow becomes laminar (dynamically stable) when $R_f > +1$.
- Statically unstable flow is, by definition, always dynamically unstable.

(2) Gradient Richardson

- A peculiar problem arises in the use of R_f; namely, we can calculate its value only for turbulent flow. We can use it to determine whether turbulent flow will become laminar but not whether laminar flow will become turbulent.

- $-\overline{w'\theta_v'} \propto \frac{\partial \overline{\theta_v}}{\partial z}$, $-\overline{u'w'} \propto \frac{\partial \overline{U}}{\partial z}$, $-\overline{v'w'} \propto \frac{\partial \overline{V}}{\partial z}$
  → basis for K-theory or eddy diffusivity theory.
- gradient Richardson # (R_i)

$$\boxed{R_i = \frac{\frac{g}{\overline{\theta_v}}\frac{\partial \overline{\theta_v}}{\partial z}}{\left[\left(\frac{\partial \overline{U}}{\partial z}\right)^2 + \left(\frac{\partial \overline{V}}{\partial z}\right)^2\right]}}$$

- dynamical stability criteria
  Laminar flow becomes turbulent when $R_i < R_c$ ← critical R_i
  Turbulent " " laminar when $R_i > R_T$ ← termination of turbulence
  $R_c = 0.21$ to 0.25
  $R_T = 1.0$

(3) Bulk Richardson

- approx $\frac{\partial \overline{\theta_v}}{\partial z} \approx \frac{\Delta \overline{\theta_v}}{\Delta z}$, $\frac{\partial \overline{U}}{\partial z} \approx \frac{\Delta \overline{U}}{\Delta z}$, $\frac{\partial \overline{V}}{\partial z} \approx \frac{\Delta \overline{V}}{\Delta z}$

$$\boxed{R_B = \frac{g\,\Delta\overline{\theta_v}\,\Delta z}{\overline{\theta_v}\left[(\Delta\overline{U})^2 + (\Delta\overline{V})^2\right]}} \longrightarrow \text{most frequently used in meteorology.}$$

- Signs are defined by $\Delta\overline{U} = \overline{U}(z_{top}) - \overline{U}(z_{bottom})$
- require artificially large value of R_c
- ✓ Richardson # itself says nothing about the intensity of turbulence, only about the yes/no presence of turbulence

(4) Examples

$$R_f = \frac{\text{buoyancy term}}{-\text{shear term}}$$

Problem: Given a fictitious SBL where $g/\overline{\theta_v} = 0.033\ ms^{-2}K^{-1}$, $\partial\overline{U}/\partial z = [u_*/(0.4z)]\ s^{-1}$, $u_* = 0.4$ m/s, and where the lapse rate, C_1, is constant with height such that there is 6°C $\overline{\theta_v}$ increase with each 200 m of altitude gained. How deep is the turbulence?

Sol) Lets use the gradient Richardson # as an indicator of dynamic stability and turbulence

$$R_i = \frac{\frac{g}{\overline{\theta_v}}\frac{\partial \overline{\theta_v}}{\partial z}}{\left(\frac{\partial \overline{U}}{\partial z}\right)^2} = \frac{\frac{g}{\overline{\theta_v}}C_1}{\left(\frac{u_*}{0.4z}\right)^2} = \frac{0.033 \times 0.03}{(0.4/0.4)^2} z^2 = (0.00099\ m^{-2})\,z^2$$

If we use $R_c = 0.25$

$$z = \sqrt{(1010\ m^2)R_c} = \sqrt{252.5\ m^2} = 15.9\ m$$

⇒ If we had used a critical termination value of $R_T = 1.0$, then we would have found a critical height of 31.8 m. Thus, below 15.9 m we expect turbulence, while above 31.8 m we expect laminar flow. Between these heights the turbulent state depends on the past history of the flow at that height. If previously turbulent it is turbulent now.

5.7 The Obukhov length (L)

- a scaling parameter that is useful in the surface layer.
- One definition of the SL is that region where turbulent fluxes vary less than 10% of their magnitude with height.
- By making the constant flux (with height) approx., one can use surface values of heat and momentum flux to define turbulence scales and nondimensionalize the TKE eq.
- $\left(-K\frac{z}{u_*^3}\right)$ * TKE eq. (section 5.1) + assume all turbulent fluxes equal their respective surface values.

$$\cdots = \underbrace{\boxed{-\frac{Kzg(\overline{w'\theta_v'})_s}{\overline{\theta_v}u_*^3}}}_{\text{III}\ \to\ \zeta} + \underbrace{\frac{Kz(\overline{u_i'u_j'})_s}{u_*^3}\frac{\partial \overline{U_i}}{\partial x_j}}_{\text{IV}} + \cdots - \underbrace{\frac{Kz\,\varepsilon|_s}{u_*^3}}_{\text{VII}}$$

where K : von Karman constant (0.35 ~ 0.42) → usually 0.4

$$\zeta = \frac{z}{L} = \frac{-Kzg(\overline{w'\theta_v'})_s}{\overline{\theta_v}u_*^3}$$

↳ Obukhov l...

$$\therefore L = \frac{-\overline{\theta_v}\, u_*^3}{kg(\overline{w'\theta_v'})_s}$$ ※ { day time : negative / night time : positive

→ Physical interpretation : It is proportional to the height above the surface at which buoyant factors first dominate over mechanical (shear) production of turbulence. For convective situations, buoyant and shear production terms are approximately equal at $z = -0.5L$.

- parameter ζ : a stability parameter → important in surface layer.
  - → Only its sign relates to static stability
    - negative : unstable
    - positive : stable (statically)
  - → a surface-layer scaling parameter

→ alternative form

$$\zeta = \frac{z}{L} = -\frac{kzw_*^3}{z_i u_*^3}$$

5.8 Dimensionless gradients

- Term IV of the dimensionless TKE eq (5.7)

→ choosing a coord. system aligned with the mean wind, assuming horizontal homogeneity, neglecting subsidence, and using $u_*^2 = -(\overline{u'w'})_0$

$$\text{Term IV} = -\frac{kz}{u_*}\frac{\partial \overline{U}}{\partial z}$$

- dimensionless wind shear, ϕ_M,

$$\phi_M = \frac{kz}{u_*}\frac{\partial \overline{U}}{\partial z}$$ → useful for studies of surface-layer wind profiles and momentum fluxes

- Similarly, dimensionless lapse rate, ϕ_H and dimensionless humidity gradient, ϕ_E

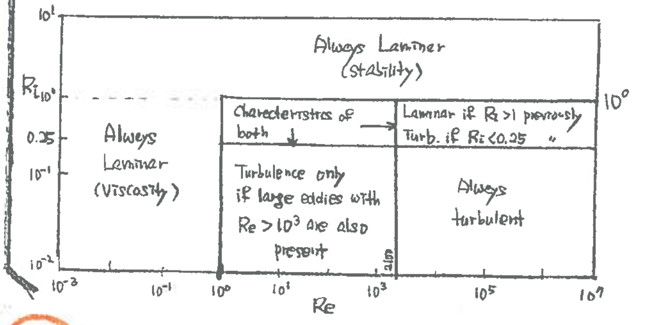
$$\phi_H = \frac{kz}{\theta_*^{SL}}\frac{\partial \overline{\theta}}{\partial z}$$

$$\phi_E = \frac{kz}{q_*^{SL}}\frac{\partial \overline{q}}{\partial z}$$

5.9 Miscellaneous Scaling parameters → 생략

5.10 Combined stability tables (static + dynamic)

Ri	−2	−1	0	1	2
Statically	unstable		stable		
Dynamically	unstable		Either (History)	stable	
Flow	Turbulent		"	Laminar	



Ch.6 Turbulence closure techniques

6.1 The closure problem

- The number of unknowns in the set of equations for turbulent flow is larger than the number of eqs.
- For any finite set of eqs, the description of turbulence is not closed.

The total statistical description of turbulence requires an infinite set of eqs. → Closure problem.

- local closure : an unknown quantity at any point in space is parameterized by values and/or gradients of known quantities at the same point

→ assume turbulence is analogous to molecular diffusion

nonlocal closure : an unknown quantity at one point is parameterized by values of known quantities at many points in space

→ assume turbulence is a superposition of eddies

6.2 Parameterization rules

- The parameterization for all unknown quantity should be physically reasonable.

It must

→ have the same dimensions as the unknown
→ have the same tensor properties
→ have the same symmetries
→ be invariant under an arbitrary transformation of coord systems.
→ " " " a Galilean (i.e inertial or Newtonian) transformation
→ satisfy the same budget eqs and constraints

6.3 Local closure – zero + half order → 생략

6.4 Local closure – first order

(1) definition

- First-order closure retains the prognostic eqs for only the zero-order mean variables such as wind, temp, and humidity.
- Consider the idealized scenario of a dry environment, horizontally homogeneous, with no subsidence. geostrophic wind is known as B.C.

→ governing eq (3.5)

$$\begin{cases} \dfrac{\partial \overline{U}}{\partial t} = f_c(\overline{V}-\overline{V_g}) - \dfrac{\partial(\overline{u'w'})}{\partial z} \\ \dfrac{\partial \overline{V}}{\partial t} = -f_c(\overline{U}-\overline{U_g}) - \dfrac{\partial(\overline{v'w'})}{\partial z} \\ \dfrac{\partial \overline{\theta}}{\partial t} = \qquad -\dfrac{\partial(\overline{w'\theta'})}{\partial z} \end{cases}$$ → unknowns $\overline{u'w'}$, $\overline{v'w'}$, $\overline{w'\theta'}$

- Let ξ be any variable, → one possible first-order closure approx. for flux $\overline{u_j'\xi'}$

$$\boxed{\overline{u_j'\xi'} = -K\frac{\partial \overline{\xi}}{\partial x_j}}$$

$m^2 s^{-1}$

⇒ gradient transport theory or K-theory

→ small-eddy closure technique (fails for large-eddy)

where K is called
- eddy viscosity
- eddy diffusivity
- eddy-transfer coefficient
- turbulent-transfer coefficient
- gradient-transfer coefficient.

K_m : eddy viscosity (for momentum)
K_H + K_E : eddy diffusivities (for heat + moisture)

(2) Analogy with viscosity

molecular stress, $\tau_{mol} = \rho\nu\dfrac{\partial \overline{U}}{\partial z}$ for a Newtonian fluid. ✓

ν → $\frac{\mu}{\rho}$

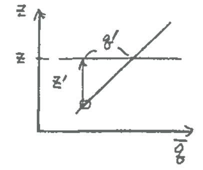
→ turbulent Reynolds stress $\tau_{Reynolds} = \boxed{\rho K_m}\dfrac{\partial \overline{U}}{\partial z}$ ✓

→ Austausch coefficient

ν : a function of the fluid
K_m : a function of the flow

(3) Mixing-length theory

- Assume that there is turbulence in a statically neutral environment, with a linear mean humidity gradient in the vertical



$$q' = -\left(\frac{\partial \overline{q}}{\partial z}\right)z'$$

- Similarly $u' = -\left(\dfrac{\partial \overline{u}}{\partial z}\right)z'$
- Let $w' = -cu'$ for $\partial\overline{U}/\partial z > 0$

$w' = cu'$ for $\partial\overline{U}/\partial z < 0$

$$\Rightarrow w' = c\left|\frac{\partial \overline{U}}{\partial z}\right|z'$$

- R (kinematic eddy flux of moisture) = $\overline{w'q'}$

10/1

$$R = -C\overline{(z')^2}\left|\frac{\partial \overline{U}}{\partial z}\right|\left(\frac{\partial \overline{\xi}}{\partial z}\right)$$

$\uparrow$ – the variance of parcel displacement distance

⇒ mixing length (l) : $l^2 = C\overline{z'^2}$

$$\therefore R = -l^2\left|\frac{\partial \overline{U}}{\partial z}\right|\left(\frac{\partial \overline{\xi}}{\partial z}\right)$$

This is directly analogous to K-theory if

$$K_E = l^2\left|\frac{\partial \overline{U}}{\partial z}\right| \longrightarrow R = -K_E\frac{\partial \overline{\xi}}{\partial z}$$

⇒ mixing-length theory tells us that the magnitude of K_E should increase as the shear increases (i.e. a measure of the intensity of turbulence) and as the mixing length increases (i.e. a measure of the ability of turbulence to cause mixing)

- In surface layer → $l^2 = k^2 z^2$

⇒ eddy viscosity in surface layer

$$K_E = k^2 z^2\left|\frac{\partial \overline{U}}{\partial z}\right|$$

⇒ Small-eddy theory

- K-theory is not recommended for use in convective mixed layers.

- Ekman pumping : The process of inducing vertical motions by BL friction

생략 ...

Ch7. Boundary conditions & Surface forcings

- effective turbulent flux : the sum of the molecular and turbulent fluxes

END