* PBL

MET? (BL): O'Brien (?, 1997)

- The log-linear profile equations for the atmospheric boundary layer are

$$v(z) = \frac{U^*}{k}\left[\ln(z/z_0) + \beta\frac{z}{L}\right] \qquad (1)$$

$$T - T_0 = \frac{T^*}{k}\left[\ln(z/z_0) + \beta\frac{z}{L}\right] \qquad (2)$$

where $$L = \frac{U^{*3}}{\frac{g}{T}\frac{H}{\rho c_p}}$$ (Monin length scale)

(a) How would you determine U^*, z_0, β, T_0, T^* from actual meteorological data? A graphical technique is adequate. Assume that you can only measure v and T on a mast. Determine H, the heat flux from

$$H = -\rho c_p K_H \frac{\partial T}{\partial z}$$

where $K_H = K_M$. Note, $k = 0.4$ (von Karman's constant).

(b) Since $\tau = \rho U^{*2}$ and $\tau = \overset{\rho}{} K_M \frac{\partial v}{\partial z}$. Derive an expression for K_M which is consistent with (1). The answer should be a profile with height of K_M containing z and constants for the particular situation. Hint: The answer does not contain v.

?

* Reynolds stress tensor & eddy viscosity

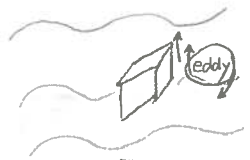
MET? : O'Brien (?)

• Define the Reynolds stress tensor. Show how it appears in equations of motion for a turbulent fluid. Define what is meant by eddy viscosity. How does eddy viscosity appear in the equations of motion for the atmosphere. What is the relationship, if any, between eddy viscosity and the Reynolds stress tensor.

Sol)

Reynolds stress exists only when the fluid is in turbulent motion. Consider a cube parcel embedded in a turbulent fluid.

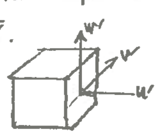
A turbulent eddy can mix air of different wind speeds into the cube of interest. When this different speed air is incorporated into one face of the cube and not the opposite face, the cube deforms because of the velocity differences between the two faces.



The rate at which air of different speeds is transported across any face of the cube is just the momentum flux. The effect of this flux on the cube is identical to what we would observe if we applied a force on the face of the cube. Thus, turbulent momentum flux acts like a stress. We call this stress due to turbulence the Reynolds stress.

In this example air moving upward (w') was mixed towards the cube at the rate u' resulting in a Reynolds stress component $-\rho\overline{u'w'}$. The magnitude of this component of the Reynolds stress or turbulent momentum flux in kinematic units is $|\overline{u'w'}|$.

When we consider just one face of the cube, air moving in any one of the 3 Cartesian directions could be mixed into it, resulting in a variety of deformations. For example we have air one face the possibility of $\overline{u'u'}$, $\overline{u'v'}$, $\overline{u'w'}$.



Since the same number of combinations hold for the other two axes, we have a total of 9 components of the Reynolds stress to account for. Note that the Reynolds stress is symmetric. That is, $\overline{u'w'} = \overline{w'u'}$.

The 9 component Reynolds stress tensor is symmetric and may be represented as

$$\begin{pmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'} \end{pmatrix} = \begin{pmatrix} ①\,\overline{u'u'} & ④\,\overline{v'u'} & ⑤\,\overline{w'u'} \\ \overline{u'v'} & ②\,\overline{v'v'} & ⑥\,\overline{w'v'} \\ \overline{u'w'} & \overline{v'w'} & ③\,\overline{w'w'} \end{pmatrix}$$

Hence we only really need to be concerned with 6 independent stress components. The Reynolds stress is a property of the flow, not of the fluid. This stress is completely described by the matrices above which contain the product of velocities (a flow characteristic) that could apply to any fluid. Such is not the case with viscous shear stresses which depend on intermolecular forces.

Although the effect of $\overline{u'w'}$ is like a stress, the Reynolds stress is not a true stress (force per unit area) as is the viscous shear stress.

• An alternative description by Holton (Sec 5.1.2)

In a turbulent fluid a field variable such as u, v, w wind components measured at a point generally fluctuate rapidly in time as eddies of various scales pass the point. In order that measurements be truely representative of the large scale flow it is necessary to average the flow over an interval of time long enough to average out the small scale eddy fluctuations but still short enough to preserve the trends in the large scale flow field. To do this we assume that the field variables can be separated into slowly varying mean fields and rapidly varying turbulent components. Following the scheme introduced by Reynolds we assume, for example,

$$u = \bar{u} + u'$$
$$w = \underbrace{\bar{w}}_{\text{mean values}} + \underbrace{w'}_{\text{turbulent fluctuations}}$$

By definition the mean of the fluctuations is zero. Further the mean of the mean is the mean $\overline{u'\bar{w}} = \overline{u'}\,\bar{\bar{w}} = \overline{u'}\,\bar{w} = 0$

The mean of the product of deviations generally does not vanish ($\overline{u'w'} \neq 0$, gener

Note that

$$\overline{u'w'} = \overline{(u-\bar{u})(w-\bar{w})} \Rightarrow \text{as such } \overline{u'w'} \text{ represents a covariance between } u \,\&\, w.$$

Finally

$$\overline{uw} = \overline{(\bar{u}+u')(\bar{w}+w')} = \overline{\bar{u}\bar{w}} + \cancel{\overline{\bar{u}w'}} + \cancel{\overline{u'\bar{w}}} + \overline{u'w'}$$

To see how terms like $\overline{u'w'}$ (a Reynolds stress) arise in the eqs of motion fo a turbulent fluid, consider the u momentum eq. (in x, y, p, t)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} - fv = -\frac{\partial \Phi}{\partial x} + \underbrace{F_x}_{\text{friction}}$$

We use the mass continuity eq.

$$\overset{ⓐ}{\frac{\partial u}{\partial x}} + \overset{ⓑ}{\frac{\partial v}{\partial y}} + \overset{ⓒ}{\frac{\partial \omega}{\partial p}} = 0$$

to rewrite the momentum eq in flux form

$$\frac{\partial u}{\partial t} + \left(u\frac{\partial u}{\partial x} + u\overset{ⓐ}{\frac{\partial u}{\partial x}}\right) + \left(v\frac{\partial u}{\partial y} + u\overset{ⓑ}{\frac{\partial v}{\partial y}}\right) + \left(\omega\frac{\partial u}{\partial p} + u\overset{ⓒ}{\frac{\partial \omega}{\partial p}}\right) - fv = -\frac{\partial \Phi}{\partial x} + F_x$$

or

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial u\omega}{\partial p} - fv = -\frac{\partial \Phi}{\partial x} + F_x$$

Apply Reynolds averaging

$$\overline{uu} = \bar{u}\bar{u} + \overline{u'u'}$$
$$\overline{uv} = \bar{u}\bar{v} + \overline{u'v'}$$
$$\overline{u\omega} = \bar{u}\bar{\omega} + \overline{u'\omega'}$$

$$\begin{cases} u = \bar{u} + u' \\ v = \bar{v} + v' \\ \omega = \bar{\omega} + \omega' \end{cases} \qquad \begin{cases} \Phi = \bar{\Phi} + \Phi' \\ F_x = \bar{F_x} + F_x' \end{cases}$$

Substitute for u, v, ω, Φ, F_x using

$$\frac{\partial\bar{u}}{\partial t}+\frac{\partial\bar{u}\bar{u}}{\partial x}+\frac{\partial\bar{u}\bar{v}}{\partial y}+\frac{\partial\bar{u}\bar{\omega}}{\partial p}-f\bar{v}+\frac{\partial u'}{\partial t}+\frac{\partial u'u'}{\partial x}+\frac{\partial u'v'}{\partial y}+\frac{\partial u'\omega'}{\partial p}-fv'$$
$$+\frac{\partial\bar{u}u'}{\partial x}+\frac{\partial u'\bar{u}}{\partial x}+\frac{\partial\bar{u}v'}{\partial y}+\frac{\partial u'\bar{v}}{\partial y}+\frac{\partial\bar{u}\omega'}{\partial p}+\frac{\partial u'\bar{\omega}}{\partial p}$$
$$=-\frac{\partial\bar{\Phi}}{\partial x}+\bar{F}_x-\frac{\partial\Phi'}{\partial x}+F_x'$$

Now average this eq. Make use of the properties of $\overline{(\ )}$ + $(\ )'$ w.r.t. $\overline{(\ )}$

All the $\overline{\bar{\alpha}\beta'}$ terms drop. All the $\overline{\alpha'}$ terms drop. All $\overline{\bar{\alpha}\bar{\beta}}=\bar{\alpha}\bar{\beta}$ and all $\overline{\alpha'\beta'}$ remain.

We are left with

$$\frac{\partial\bar{u}}{\partial t}+\frac{\partial\bar{u}\bar{u}}{\partial x}+\frac{\partial\bar{u}\bar{v}}{\partial y}+\frac{\partial\bar{u}\bar{\omega}}{\partial p}-f\bar{v}=-\frac{\partial\bar{\Phi}}{\partial x}+\bar{F}_x$$
$$+\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'\omega'}}{\partial p}$$

→ these are turbulent fluxes of momentum (u-momentum)

we call $-\rho\overline{u'u'}$, $-\rho\overline{u'v'}$, $-\rho\overline{u'\omega'}$ Reynolds stresses

Summary

To obtain Reynolds stresses in governing eqs

① expand variables as $\alpha=\bar{\alpha}+\alpha'$

② substitute into governing eqs.

③ take mean of governing eqs remembering that

ⓐ $\overline{\alpha'}=0$, ⓑ $\overline{\bar{\alpha}}=\bar{\alpha}$, ⓒ $\overline{\bar{\alpha}\bar{\beta}}=\bar{\alpha}\bar{\beta}$

ⓓ $\overline{\bar{\alpha}\beta'}=0$. ⓔ $\overline{\alpha'\beta'}\neq 0$ (generally)

Note that by performing this Reynolds decomposition that we have introduced new variables into our system of eqs. (turbulent fluxes). To solve the system of governing eqs. Closure assumptions must be made to approx the unknown fluxes. One approach for achieving this end leads to the concept of an eddy viscosity.

• Section 5.3.2 of Holton.

The traditional approach to this closure problem is to assume that turbulent eddies act in a manner analogous to molecular diffusion. Under this assumption the flux of a given field is proportional to the local gradient of the mean. We call the proportionality constant the eddy viscosity. This type of closure is called K-theory. As an example, the Reynolds stress term $\overline{u'w'}$ is approximated as

$$\overline{u'w'}=-K_m\frac{\partial\bar{u}}{\partial z}$$

where $K_m\equiv$ eddy viscosity coefficient.

Limitations of K-theory

• eddy viscosities depend on the flow rather than the physical properties of the fluid (as for the molecular viscosity coefficient) ∴ eddy viscosity coefficients must be determined empirically for each one.

• more energetic eddies have scales too large to assume that $\overline{u'w'}$ is proportional to the local vertical gradient of the mean (as the eddies are of the same scale as the gradient)

Mixing length hypothesis of Prandtl.

① A parcel of fluid that is displaced vertically will carry the mean properties of its original level for a characteristic distance ℓ' and then well mix with its surroundings. This is similar to the molecular case where a molecule travels a distance d', the mean free path, before colliding and exchanging momentum with another molecule. By further analogy to the molecular case, the parcel displacement is ②assumed to create a turbulent fluctuation whose magnitude depends on ℓ' and the gradient of the mean property. For example,

$$u'=-\ell'\frac{\partial\bar{u}}{\partial z}$$

③ We assume the turbulent motions are isotropic so that

$$w'=+\ell'\left|\frac{\partial\bar{V}}{\partial z}\right|$$

With these two expressions

$$-\overline{u'w'}=\overline{(-)\left(-\ell'\frac{\partial\bar{u}}{\partial z}\right)\left(\ell'\left|\frac{\partial\bar{V}}{\partial z}\right|\right)}$$
$$-\overline{u'w'}=\left(\overline{(\ell')^2}\left|\frac{\partial\bar{V}}{\partial z}\right|\right)\frac{\partial\bar{u}}{\partial z}$$
$$-\overline{u'w'}=K_m\frac{\partial\bar{u}}{\partial z}$$

→ the eddy viscosity defined as

$$K_m=\overline{(\ell')^2}\left|\frac{\partial\bar{V}}{\partial z}\right|$$

→ this definition allows for eddies of greater size to have a greater effect.

* Barotropic dynamics (Shallow water eq. → domain invariants + potential vorticity)

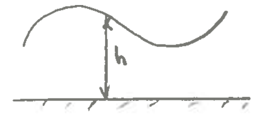
MET? (Barotropic Dynamics): O'Brien (1 hour, 1997) **

• Consider a restricted class of motions of a homogeneous inviscid fluid for which the equations of motion are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv+g\frac{\partial h}{\partial x}=0 \qquad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu+g\frac{\partial h}{\partial y}=0 \qquad (2)$$

$$\frac{\partial h}{\partial t}+\frac{\partial uh}{\partial x}+\frac{\partial vh}{\partial y}=0 \qquad (3)$$



The symbols have the usual meaning. The fluid is bounded at the bottom by a rigid surface and the top is a free surface.

(a) Show that if the fluid is bounded by rigid vertical walls that E is invariant for this system. What is E physically?

$$E=\iint_R\left[\frac{1}{2}\left(u^2+v^2\right)h+\frac{1}{2}gh^2\right]dxdy$$

where R is the region bounded by the fixed lateral boundaries.

(b) Show that if ζ is the vertical component of relative vorticity that

$$\frac{\zeta+f}{h}=\text{constant along a trajectory.}$$

What do we call this theorem?

(a)

From $u*(1)$ and $v*(2)$

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)+u\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)+v\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)-fuv+gu\frac{\partial h}{\partial x}=0$$

$$+)\ \frac{\partial}{\partial t}\left(\frac{v^2}{2}\right)+u\frac{\partial}{\partial x}\left(\frac{v^2}{2}\right)+v\frac{\partial}{\partial y}\left(\frac{v^2}{2}\right)+fuv+gv\frac{\partial h}{\partial y}=0$$

$$\frac{\partial}{\partial t}\left(\frac{u^2+v^2}{2}\right)+u\frac{\partial}{\partial x}\left(\frac{u^2+v^2}{2}\right)+v\frac{\partial}{\partial y}\left(\frac{u^2+v^2}{2}\right)+u\frac{\partial}{\partial x}(gh)+v\frac{\partial}{\partial y}(gh)=0$$

Define, $K=\frac{1}{2}(u^2+v^2)$. The above may be rewritten

$\underline{V}_H=u\hat{i}+v\hat{j}$

$$\frac{\partial K}{\partial t}+\underline{V}_H\cdot\nabla K+\underline{V}_H\cdot\nabla gh=0 \qquad (4)$$

Now let's work with (3). Expand (3) as

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=-h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

If we look at E we see the integrand is of the form $Kh+\frac{gh^2}{2}$ or $(K+\frac{gh}{2})h$. Let's multiply the above eq by $(K+gh)$. It will become apparent later why we use gh and not $gh/2$

$$(K+gh)\frac{\partial h}{\partial t}+u(K+gh)\frac{\partial h}{\partial x}+v(K+gh)\frac{\partial h}{\partial y}=-h(K+gh)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

✓ $$K\frac{\partial h}{\partial t}+gh\frac{\partial h}{\partial t}+\underline{V}_H\cdot(K+gh)\nabla h=-h(K+gh)(\nabla\cdot\underline{V}_H) \qquad (5)$$

We multiply (4) by h

✓ $$h\frac{\partial K}{\partial t}+\underline{V}_H\cdot h\nabla K+\underline{V}_H\cdot h\nabla gh=0 \qquad (6)$$

add (5) + (6)

$$\left(K\frac{\partial h}{\partial t}+h\frac{\partial K}{\partial t}\right)+\frac{\partial}{\partial t}\left(\frac{gh^2}{2}\right)+\underline{V}_H\cdot(K+gh)\nabla h+\underline{V}_H\cdot h\nabla K+\underline{V}_H\cdot h\nabla gh$$

$$=-h(K+gh)(\nabla\cdot\underline{V}_H)$$

$$\frac{\partial Kh}{\partial t}+\frac{\partial}{\partial t}\left(\frac{gh^2}{2}\right)+\underline{V}_H\cdot(K\nabla h+h\nabla K)+\underline{V}_H\cdot(gh\nabla h+h\nabla gh)$$

$$=-Kh\nabla\cdot\underline{V}_H-gh^2(\nabla\cdot\underline{V}_H)$$

$$\frac{\partial}{\partial t}\left(Kh+\frac{gh^2}{2}\right)+\underline{V}_H\cdot\nabla Kh+Kh\nabla\cdot\underline{V}_H+\underline{V}_H\cdot\nabla gh^2+gh^2\nabla\cdot\underline{V}_H=0$$

$$\frac{\partial}{\partial t}\left(Kh+\frac{gh^2}{2}\right)+\nabla\cdot Kh\underline{V}_H+\nabla\cdot gh^2\underline{V}_H=0$$

$$\frac{\partial}{\partial t}\left(Kh+\frac{gh^2}{2}\right)+\nabla\cdot(Kh+gh^2)\underline{V}_H=0 \qquad (7)$$

Integrate (7) over the region R with rigid lateral boundaries.

$$\iint_R\frac{\partial}{\partial t}\left(Kh+\frac{gh^2}{2}\right)dxdy=-\iint_R\nabla\cdot(Kh+gh^2)\underline{V}_H\,dxdy$$

Since the walls are rigid there is no net flux in or out of the domain.

Thus, $\iint_R\nabla\cdot(Kh+gh^2)\underline{V}_H\,dxdy=0$

and $\frac{\partial}{\partial t}\iint_R\left(Kh+\frac{gh^2}{2}\right)dxdy=0$

This implies that

$$\iint_R\left(Kh+\frac{gh^2}{2}\right)dxdy=E \qquad (8)$$

where E is some constant

Physically E represents the total energy in the fluid system within the region R.

$(K+\frac{1}{2}gh)h$

$\frac{1}{2}gh \rightarrow$ potential energy / mass

$K \rightarrow$ kinetic energy / mass

(b)

We call $\frac{\zeta+f}{h}$ the potential vorticity. The trajectory is the path an object follows moving through space. The total derivative $\frac{d\alpha}{dt}$ represents the time rate of change of α (some scalar for instance) following the motion. That is, $\frac{d\alpha}{dt}$ is measured along a trajectory. Thus, to show that potential vorticity $\frac{\zeta+f}{h}$ is constant (conserved) along a trajectory we must show

$$\frac{d}{dt}\left(\frac{\zeta+f}{h}\right)=0$$

The first step is to derive a vorticity eq.

Note that

$$\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V}_H = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

Therefore, form

$$\frac{\partial}{\partial x}(2) \Rightarrow \frac{\partial}{\partial t}\frac{\partial v}{\partial x}+u\frac{\partial}{\partial x}\frac{\partial v}{\partial x}+v\frac{\partial}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+f\frac{\partial u}{\partial x}+g\frac{\partial}{\partial x}\frac{\partial h}{\partial y}=0$$

$$-\frac{\partial}{\partial y}(1) \Rightarrow \frac{\partial}{\partial t}\left(-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+f\frac{\partial v}{\partial y}+\frac{\partial f}{\partial y}v$$

$$-g\frac{\partial}{\partial y}\frac{\partial h}{\partial x}=0$$

$$\frac{\partial\zeta}{\partial t}+\mathbf{V}_H\cdot\nabla\zeta+\zeta\nabla\cdot\mathbf{V}_H+f\nabla\cdot\mathbf{V}_H+\mathbf{V}_H\cdot\nabla f+0=0$$

Note that f is independent of time as $\frac{\partial\zeta}{\partial t}=\frac{\partial}{\partial t}(\zeta+f)$

$$\frac{\partial}{\partial t}(\zeta+f)+\mathbf{V}_H\cdot\nabla(\zeta+f)+(\zeta+f)\nabla\cdot\mathbf{V}_H=0 \qquad (9)$$

From the continuity eq (3)

$$\frac{\partial h}{\partial t}+\mathbf{V}_H\cdot\nabla h+h\nabla\cdot\mathbf{V}_H=0$$

$$\downarrow$$

$$\frac{dh}{dt}=-h\nabla\cdot\mathbf{V}_H \longrightarrow \frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{V}_H\cdot\nabla$$

$$\underline{\frac{1}{h}\frac{dh}{dt}=-\nabla\cdot\mathbf{V}_H} \qquad (10)$$

Rewrite (9) as

$$\left(\frac{1}{\zeta+f}\right)\frac{d}{dt}(\zeta+f)=-\nabla\cdot\mathbf{V}_H \qquad (11)$$

Compare (10) & (11), we see

$$\frac{1}{h}\frac{dh}{dt}=\frac{1}{\zeta+f}\frac{d}{dt}(\zeta+f)$$

$$\frac{d\ln h}{dt}=\frac{d}{dt}\ln(\zeta+f)$$

$$\frac{d}{dt}\ln(\zeta+f)-\frac{d}{dt}\ln h=0$$

$$\frac{d}{dt}\ln\left(\frac{\zeta+f}{h}\right)=0$$

Or $\frac{d}{dt}\left(\frac{\zeta+f}{h}\right)=0$

$\uparrow$ potential vorticity is conserved along the trajectory.

* aerodynamic formulae + drag coef. (Surface fluxes)

OCP5551

MET? (Air-Sea): O'Brien (?)

• Question

(a) Define in fundamental quantities the fluxes of momentum, heat, and moisture in math symbols and completely explain each symbol.

(b) Define the 3 fluxes in (a) as bulk aerodynamic formulae, i.e., with drag coefficients.

✓ (c) Draw a sketch of the wind profile in neutral, stable and unstable conditions in the lowest 30 m over the sea. Label each of the 3 curves.

(d) Describe how drag coefficient vary stability and height.

(a)

fundamental quantities

- $Q \equiv$ physical quantity which is being "fluxed" → Q units per unit mass
- $w \equiv$ vertical volume flux
- flux ≡ transport per unit area per unit time
- $\rho w \equiv$ vertical mass flux
- $\rho Q w \equiv$ vertical flux of Q

Specifically

- momentum flux $F_m = -\rho u w$ → this is $\frac{\text{force}}{\text{area}}$ → Pressure $\left(\frac{N}{m^2}\right) = P_a$
  - → units of u are $\frac{\text{momentum}}{\text{unit mass}}$
- Sensible heat flux $F_H = \rho (C_p T) w$ → this is $\frac{\text{Power}}{\text{Area}} = \frac{J}{m^2 \cdot s} = \left(\frac{W}{m^2}\right)$
  - → $C_p T \equiv$ Joules/kg
- latent heat flux $F_q = \rho (L q) w$ → this is $\frac{\text{Power}}{\text{area}} = \left(\frac{W}{m^2}\right)$
  - → $Lq \equiv$ Joules/kg

If we decompose $Q = \overline{Q} + Q'$
- Q' → eddy or turbulent fluctuation from mean
- $\overline{Q}$ → mean or large scale value

then we can rewrite the above in terms of eddy fluxes.

Note

$$(\overline{\alpha}+\alpha')(\overline{\beta}+\beta') = \overline{\alpha}\overline{\beta} + \cancel{\overline{\overline{\alpha}\beta'}}^{\,0} + \cancel{\overline{\alpha'\overline{\beta}}}^{\,0} + \overline{\alpha'\beta'} \quad \Leftarrow \text{Reynolds averaging}$$

$$= \overline{\alpha}\overline{\beta} + \underbrace{\overline{\alpha'\beta'}}_{\text{eddy term}}$$

In terms of eddy fluxes

- $F_m = -\rho\overline{u'w'}$: momentum flux in PBL is downward (a sink)
  - → eddy flux of eddy momentum
- $F_H = \rho C_p \overline{T'w'}$: sensible heat flux in PBL is upward (source)
  - → eddy flux of eddy sensible heat
- $F_q = \rho L\overline{q'w'}$: latent heat flux in PBL is upward (source)
  - → eddy flux of eddy latent heat

Note that the quantity $\overline{\alpha'\beta'}$ is proportional to the correlation between α and β. More precisely,

✓ $$r_{\alpha\beta} = \frac{\overline{\alpha'\beta'}}{\sigma_\alpha \sigma_\beta}$$

- σ_β → (variance)$^{1/2}$ of β
- σ_α → " " α
- $r_{\alpha\beta}$ → correlation coefficient between α + β $\in [-1, 1]$

This relationship between eddy terms $\overline{\alpha'\beta'}$ (a covariance between α + β) and the correlation lead to the correlation method for estimating fluxes.

Via this method,

$$F_m = -\rho\overline{u'w'} = -\rho\, r_{uw}\sigma_u\sigma_w$$
$$F_H = \rho C_p\overline{T'w'} = \rho\, r_{Tw}\sigma_T\sigma_w$$
$$F_q = \rho L\overline{q'w'} = \rho\, r_{qw}\sigma_q\sigma_w$$

eddy correlation method
- + involves fewest assumptions as to the nature of turbulence
- − direct measurements of eddy fluctuations are difficult and so usually must rely on relationship between mean + eddy variables

(b)

• Bulk aerodynamic formulae

$$F_m = -(\rho C_D \overline{u}_a)(\overline{u}_a)$$
$$F_H = C_p(\rho C_D \overline{u}_a)(\overline{T}_s - \overline{T}_a)$$
$$F_q = L(\rho C_D \overline{u}_a)(\overline{q}_s - \overline{q}_a)$$

C_D → drag coefficient.

$\overline{T}_a, \overline{q}_a, \overline{u}_a$ ~10m, anemometer level

$\overline{T}_s, \overline{q}_s$ sfc

To see where these formulae come from we proceed as in Dr. LaSeur's Tropical II notes.

The mixing length approach (K theory) is a simple method of relating the eddy fluctuations to the average values. It may be viewed to some extent as an analog to kinetic theory of gases employed on the molecular scale. The mixing length approach assumes that eddy fluctuations are given by

✓ $$Q' = \ell\frac{\partial\overline{Q}}{\partial z}$$

Where $\ell \equiv$ mixing length is a vertical distance over which the initial value of Q is conserved before the eddy mixes with the environmental mean at a reference level.

Thus, $u' = \ell\frac{\partial\overline{u}}{\partial z}$, $T' = \ell\frac{\partial\overline{T}}{\partial z}$, $q' = \ell\frac{\partial\overline{q}}{\partial z}$

It is further assumed that the eddy fluctuations are isotropic in the sense that $w' = u'$. With these assumptions

$$F_m = -\rho\overline{u'w'} = -\rho\left[-\overline{\ell^2}\left(\frac{\partial\overline{u}}{\partial z}\right)^2\right]$$

Define $\boxed{K_m = \overline{\ell^2}\left|\frac{\partial\overline{u}}{\partial z}\right|}$ as an eddy viscosity (units of m^2/s)

Then

✓ $$F_m = \rho K_m \frac{\partial\overline{u}}{\partial z}$$

Note that $\overline{\ell^2}$ is the variance of the mixing length ℓ.

Similarly

✓ $$F_H = -\rho C_p K_H \frac{\partial\overline{T}}{\partial z}$$

✓ $$F_q = -\rho L K_q \frac{\partial\overline{q}}{\partial z}$$

We usually assume $K_m \simeq K_H \simeq K_q$. This is most nearly valid for neutral stratification. This condition is often observed over the tropical oceans.

A further assumption as to the nature of $(\overline{\ell^2})^{1/2} = \sigma_\ell$ is that in the sfc layer of a few tens of meters

✓ $$\boxed{(\overline{\ell^2})^{1/2} = \sigma_\ell = kz}$$

where $k \equiv$ Von Karman constant ~ 0.4. The above approx implies that the rms mixing length is proportional to the height above the sfc in the lowest portion of the PBL.

We define the friction velocity u^* such that

$$u^* = \left(\frac{\tau}{\rho}\right)^{1/2} \quad \text{where } \tau = \rho K_m \frac{\partial \bar{u}}{\partial z} = F_m$$
$$\hookrightarrow = \rho u_*^2$$

Therefore we can write

$$\tau = \rho u_*^2$$
$$\rho K_m \frac{\partial \bar{u}}{\partial z} = \rho u_*^2 \longrightarrow l^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2 = u_*^2$$
$$\frac{\partial \bar{u}}{\partial z} = \left(\frac{u_*^2}{l^2}\right)^{1/2} = \frac{u_*}{kz}$$

We can integrate $\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{kz}$ from z_0 where $u=0$ (near the sfc) to z

$$\int_{z_0}^{z} \frac{\partial \bar{u}}{\partial z} dz = \int_{z_0}^{z} \frac{u_*}{kz} dz$$
$$\bar{u}(z) - \cancel{\bar{u}(z_0)}^{\,0} = \frac{u_*}{k} \ln\left(\frac{z}{z_0}\right)$$

or ✓ $\boxed{\bar{u}(z) = \frac{u_*}{k} \ln\left(\frac{z}{z_0}\right)}$

This is the so-called logarithmic wind profile. The height z_0 is the "roughness" height. It is the height at which $\bar{u}$ is assumed to be zero.

This surface layer or constant flux layer is also called the Prandtl layer. With the assumptions used, τ, F_H, F_g, u_* are all constant in the sfc layer and are assumed to equal surface values. Thus, if the mean values are known in this layer and values of k are determined (or assumed), the sfc stress and heat fluxes can be determined or parameterized. The depth of the Prandtl layer is a few tens of meters.

The derivatives of the mean values in the "K" theory expressions should require at least two levels of measurement for their finite difference approx. A simpler method that requires only one level of data can be developed with the use of the logarithmic wind law.

Thus

$$\tau = \rho u_*^2 = \rho \underbrace{\left(\frac{k}{\ln\left(\frac{z}{z_0}\right)}\right)^2}_{\text{from logarithmic wind profile}} \bar{u}^2$$

$$u^* = k\bar{u}\left[\ln\left(\frac{z}{z_0}\right)\right]^{-1}$$

Now define a drag coefficient C_D at a reference level $z = z_a$ (anemometer level ~10m), so that

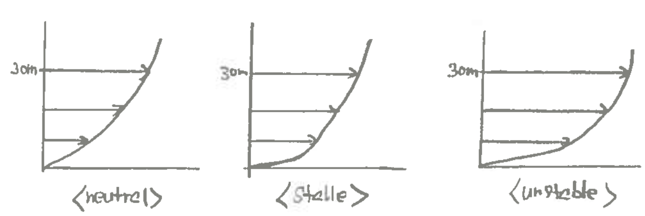
$$\boxed{C_D = \left(\frac{k}{\ln\left(\frac{z}{z_0}\right)}\right)^2}$$

and $\tau = \rho C_D \bar{u}_a^2 = (\rho C_D \bar{u}_a)\bar{u}_a$

By analogy

$$F_H = C_p(\rho C_D \bar{u}_a)(\bar{T}_s - \bar{T}_a)$$
$$F_g = L(\rho C_D \bar{u}_a)(\bar{g}_s - \bar{g}_a)$$

(c)



(d)

We have defined the drag coefficient

$$C_D = k^2 / \left[\ln\left(\frac{z_a}{z_0}\right)\right]^2 \quad \because k = \text{Von Karman constant} \sim 0.4$$

The drag coefficient depends on the roughness of the sfc and the wind speed. With a rough sfc and strong wind, the value of C_D decreases with height.

- as height $z_a \uparrow$, $C_D \downarrow$
- as aerodynamic roughness $z_0 \uparrow$, $C_D \uparrow$

→ In stable situation we expect smaller C_D since the effect of sfc is confined to shallow sfc layer.

→ In unstable case we expect larger C_D since the effect of sfc is "felt" further above sfc due to vertical motion.

→ In all cases C_D decreases with height ⇒ effect of sfc drag decreases with height.

* bulk aerodynamic formula

OCP5551

MET? (Air-Sea): O'Brien (1 hour, 1995)

- Write down the bulk aerodynamic formula for estimating the wind stress, sensible heat flux and evaporation from the ocean.

1. Define all variables.
2. ✓ What are typical values in MKS of all the variables and parameters in summer at 30°N?
   - a: Which variables can be determined from satellites? And with what accuracy? And with what measurement techniques. [Note: all cannot!] ?
   - b: ✓ These fluxes are defined using concepts from turbulence theory. Chose one of the three fluxes and discuss the problems encountered using a direct (turbulence) method.

see the previous question ? + next question

$$\begin{cases} \overline{F}_m = +^{?} \rho C_D \overline{u}_a^2 \\ \overline{F}_H = C_p \rho C_D \overline{u}_a (\overline{T}_s - \overline{T}_a) \\ \overline{F}_q = L \rho C_D \overline{u}_a (\overline{q}_s - \overline{q}_a) \end{cases}$$

* bulk aerodynamic formulas

OCP5551

MET? (Air-Sea): O'Brien (30 minutes) * , 1987

- The most widely used formulas to estimate the fluxes between the ocean and the atmosphere are the bulk aerodynamic formulas which involve drag coefficients.
  - a) Write down formulas for the wind stress, sensible heat flux and the latent heat flux. Identify all symbols.
  - ✓ b) What is the order of magnitude of the drag coefficients and the fluxes for 20 knot winds in mid-latitudes? Assume any reasonable value for the unknown terms. Please use MKS units.
  - c) The drag coefficients depend on wind speed and atmospheric stability. Briefly explain the functional dependence.

* See the previous (2) question.

a)

Sfc stress

$\tau_0 = -\rho_0 \overline{u'w'}$

← units of N/m^2

where, $\tau_0 \equiv$ surface stress

$\rho_0 \equiv$ density of air at sfc.

$u' = u - \overline{u} \equiv$ turbulently fluctuating part of the wind in the x direction.

$w' = w - \overline{w} \equiv$ " " " " " " " z "

$\overline{(\ )}$: space and/or time mean to resolve mean flow characteristics → what we measure.

Sensible heat flux

$H_0 = \rho_0 C_p \overline{w'\theta'}$

← W/m^2

where, $H_0 \equiv$ sfc sensible heat flux

$C_p \equiv$ specific heat at constant pressure ($1004\ J/kg\,K$)

$\theta' = \theta - \overline{\theta} \equiv$ turbulently fluctuating part of potential temp.

Latent heat flux

$Q_0 = \rho_0 L \overline{w'q'}$

← W/m^2

(Note that $\frac{Q_0}{L} = E$ → evaporation rate
← units of $\frac{kg}{m^2}\frac{1}{s}$
(multiply by $\frac{1}{\rho}$ to get units of m/s))

where, $Q_0 \equiv$ Latent heat flux

$L \equiv$ latent heat of vaporization for water ($\sim 2.5\times10^6\ J/kg$)

$q' = q - \overline{q} \equiv$ turbulently fluctuating part of specific humidity.

⇒ fluxes using the drag coefficient.

Wind stress

$\tau_0 = \rho_0 C_m (\overline{U}_{10} - U_s)^2$

where, $\rho_0 \equiv$ air density at sfc

$C_m \equiv$ drag coefficient for momentum flux

$\overline{U}_{10} \equiv$ mean wind speed at 10 m

$U_s \equiv$ speed of the air-sea interface at the air-sea interface.

Sensible heat flux

$H_0 = \rho_0 C_p C_\theta (\overline{U}_{10} - U_s)(T_s - \overline{T}_{10})$

where, $C_p \equiv$ specific heat of air at constant pressure.

$C_\theta \equiv$ drag coefficient for heat flux

$T_s \equiv$ skin temp of air-sea interface.

$\overline{T}_{10} \equiv$ mean temp at 10 m.

Latent heat flux

$Q_0 = \rho_0 L C_q (\overline{U}_{10} - U_s)(q_s - \overline{q}_{10})$

where, $L \equiv$ latent heat of water vaporization

$C_q \equiv$ drag coeff. for moisture flux

$q_s \equiv$ specific humidity of the air-sea interface which is saturated.

$\overline{q}_{10} \equiv$ mean specific humidity at 10m

b)

Typical values of $C_m \sim 2\times10^{-3}$

$\left.\begin{matrix} C_\theta \\ C_q \end{matrix}\right) \sim 1.2\times10^{-3}$ ✓

We define the Bowen ratio as the ratio of sensible to latent heat

$$\beta = \frac{H_0}{Q_0}$$

over land $\beta \sim$ order 1

tropical oceans $\beta \sim$ order 0.1 → latent heat is an order of magnitude greater than sensible heat flux.

Suppose we are in the mid-lats

Take $\overline{U}_{10} \sim 10\ m/s$ $\quad T_s - \overline{T}_{10} \sim 4K$ $\quad \rho_0 \sim 1.25\ kg/m^3$

$U_s \sim 0$ $\quad q_s - \overline{q}_{10} \sim 2g/kg$

use $C_m = 2\times10^{-3}$; $C_\theta = C_q = 1.2\times10^{-3}$

$C_p = 1004 \frac{J}{kg\,K}$; $L = 2.5\times10^6\ J/kg$

Then estimates of the wind stress, sensible heat flux and latent heat flux are as follows

Wind stress

$\tau_0 = \rho_0 C_m (\overline{U}_{10} - U_s)^2 = (1.25\ kg/m^3)(2\times10^{-3})(10\ m/s)^2 \approx 0.1\ Pa$.

$\times \left(\frac{kg\,m}{s^2} = N,\ \frac{Nm}{m^3} = Pa\right)$

Sensible heat

$H_0 = \rho_0 C_p C_\theta (\overline{U}_{10} - U_s)(T_s - \overline{T}_{10})$

$= (1.25 \frac{kg}{m^3})(1004 \frac{J}{kg\,K})(1.2\times10^{-3})(10\ m/s)(4K)$

$\approx 60\ W/m^2$

latent heat flux

$Q_0 = \rho_0 L C_q (\overline{U}_{10} - U_s)(q_s - \overline{q}_{10})$

$= (1.25 \frac{kg}{m^3})(2.5\times10^6 \frac{J}{kg})(1.2\times10^{-3})(10\ m/s)(2\times10^{-3})$

$\approx 75\ W/m^2$

⇒ Bowen ratio

$$\beta = \frac{H_0}{Q_0} \sim 0.8$$

mid-lats.

Repeat with $\overline{U}_{10} = 5\ m/s$

$\tau \approx 0.1\ pa$

$F_H \approx 30.12\ W/m^2$

$F_L \approx 37.5\ W/m^2$

→ $\beta = 0.8$

tropics

Repeat with $\overline{U}_{10} = 5\ m/s$, $\Delta T = 2K$, $\Delta q = 4\ g/kg$

$F_H \sim 15\ W/m^2$

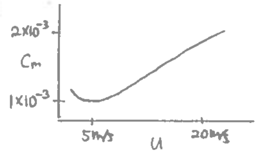
$F_L \sim 75\ W/m^2$

→ $\beta \sim 0.2$

c)

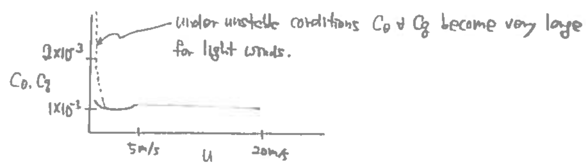
Wind speed (assume near neutral conditions)

C_m : Generally, as the wind speed increases so does the drag coefficient increase. This is true with regards to the drag coefficient for momentum.



There is a slight (very slight) tendency for C_m to increase as wind speed decreases at very low wind speeds. This effect is more pronounced for C_θ and C_q.

C_θ and C_q : In contrast to C_m, the exchange coefficients for sensible & latent heat appear to be independent of wind speed for the range of ~5 m/s to 20 m/s. Above 20 m/s there is some indication that C_q increases due to evaporation of sea spray while C_θ decreases due to evaporation of the sea spray.



Height dependence

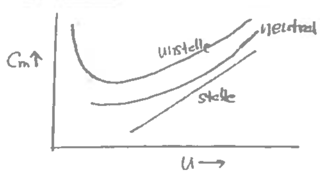
We've shown

$$C_D = k^2\left(\ln\left(\frac{z}{z_0}\right)\right)^{-2}$$

in the sfc layer. Thus the drag coefficients generally decrease as height above the sfc increases. Physically this is reasonable as we would expect the sfc to have a decreasing impact on the atmosphere as we move away from the sfc.

Stability dependence

The relationship of the drag coefficients to stability is very complex. In simplistic terms, however, as stability increases there is a damping of turbulent motions so we expect the drag coefficients to decrease



*drag coeff. → log-wind profile

OCP5551

MET? (Air-Sea): O'Brien (30 minutes) *

✓ • Suppose a meteorologist is able to estimate the drag coefficient, C_D, at a height of 2 m over the ocean during slightly unstable conditions. How would he adjust the values to the standard height at 10 m? Hint: derive a formula based on log-linear law.

* see Haltiner & Williams NWP pp267–274 or Holton, section 5.3.5

Sol)

We start with a derivation of the log-wind profile. For the sake of simplicity, we will assume that the wind in the sfc layer is along the x axis.

For a surface layer of neutral static stability, the mixing length l' may be approx by a linear function as the distance from the sfc. That is,

$$l' = kz$$

where $k \equiv$ Von Karman constant ~ 0.4 (determined empirically)

According to flux gradient (K) theory with the mixing length hypothesis made, the turbulent momentum flux or stress

$$\tau = -\rho\overline{u'w'} = \rho K_M \frac{\partial \bar{u}}{\partial z}$$

(× viscous force per unit area) ↳ shearing stress $\tau_{zx} = \mu\frac{\partial u}{\partial z}$

where $K_M = \overline{(l')^2}\left|\frac{\partial \bar{V}}{\partial z}\right|$ is the eddy viscosity.

We define a friction velocity u_* such that

$$u_*^2 \equiv |\overline{(u'w')_s}|$$

Measurements indicate that the momentum flux near the sfc is of order $0.1\ \frac{m^2}{s^2}$

so that $u_* \simeq (0.1\ m^2/s^2)^{1/2} = 0.3$ m/s

Note that the sfc eddy stress is equal to

$$\tau_s = \rho_0 u_*^2$$

In the sfc layer (lowest 50 m above sfc, typically) we may take

$$-\rho\overline{u'w'} = \rho K_M \frac{\partial \bar{u}}{\partial z}$$

$$u_*^2 = |\overline{u'w'}|$$

to obtain

$$+\rho u_*^2 = \rho K_M \frac{\partial \bar{u}}{\partial z} \longrightarrow u_*^2 = K_M \frac{\partial \bar{u}}{\partial z}$$

In the sfc layer we assume $l' = kz$ so

$$K_M = \overline{(l')^2}\frac{\partial \bar{u}}{\partial z} \text{ becomes } K_M = k^2 z^2 \frac{\partial \bar{u}}{\partial z}$$

and so

$$u_*^2 = (kz)^2 \frac{\partial \bar{u}}{\partial z}\frac{\partial \bar{u}}{\partial z}$$

or

$$u_* = kz\frac{\partial \bar{u}}{\partial z}$$

$$\frac{u_*}{kz} = \frac{\partial \bar{u}}{\partial z}$$

Integrate from z_0 to z $(z_0 < z)$

$$\boxed{\frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) = u(z) - u(z_0)}$$

Select z_0 such that $u(z_0) = 0$

We have the log wind profile for a neutral sfc layer.

$$☆\ \boxed{u(z) = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right)}$$

$$\frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) = u(z) - u(z_0)$$

↳ $u(z_0)$ is no longer assumed to be zero.

Repeat the above analysis *

$$\tau = \rho u_*^2 \text{ and } \tau = \rho C_D[u(z)]^2$$

$$\boxed{u_*^2 = C_D[u(z)]^2}$$

Substitute for $u(z)$ using $u(z) = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) + u(z_0)$

Then

$$\boxed{u_*^2 = C_D\left[\frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) + u(z_0)\right]^2}$$

• Suppose we know $C_D(z=2m)$. How do we estimate C_D at $z=10m$.

① From $u_*^2 = C_D[u(z)]^2$ we know

$$u(z) = \left[\frac{u_*^2}{C_D}\right]^{1/2}$$

So given $C_D(z=2)$ we can estimate $u(z)$ using a reasonable value for $u_*^2 \sim 0.1$ m²/s²

② From $C_D(z) = \left(u_* \Big/ \left[\frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) + u(z_0)\right]\right)^2$

We estimate C_D at $z = 10m$ assuming reasonable values for $u_*^2 \sim 0.1$ m²/s², $k \sim 0.4$ and setting $z_0 = 2m$ and using $u(2m)$ from ①

• A different way to parameterize the stress in the sfc layer is to use bulk aerodynamic. Here we assume there exists a drag coeff., C_D, such that

$$✓\ \vec{\tau}_s = \rho_s C_D V_s \vec{V}_s$$

Surface stress vector (is proportional to) the surface layer wind speed and points in the direction of this wind.

The drag coefficient is the proportionality constant.

As applied to our example where the surface layer stress is parallel to the x axis the above becomes

$$✓\ \tau = \rho C_D [u(z)]^2$$

We use the expression for $u(z)$, the log wind profile

$$\tau = \rho C_D\left[\frac{u_*}{k}\ln\left(\frac{z}{z_0}\right)\right]^2$$

This is ρu_*^2

$$\cancel{\rho} u_*^{\cancel{2}} = \cancel{\rho} C_D \frac{u_*^{\cancel{2}}}{k^2}\left(\ln\left(\frac{z}{z_0}\right)\right)^2$$

Solve for C_D

$$☆\ C_D(z) = k^2\left[\ln\left(\frac{z}{z_0}\right)\right]^{-2}$$

$$* \quad \tau = -\rho\overline{u'w'} = \rho K_M \frac{\partial \bar{u}}{\partial z} = \rho u_*^2 = \rho C_D [u(z)]^2$$

$K_M = \overline{(l')^2}\left|\frac{\partial \bar{V}}{\partial z}\right|$