## 16.1 Mathematical Definition of Dispersion:

Consider two waves of slightly different phase velocities, $c_1$ and $c_2$, and wave numbers, $k_1$ and $k_2$, propagating in an elastice material. Figure 16.1 shows the conditions under which wave shapes can remain constant during propagation. Consider the amplitudes of each wave as sinusoids:

$$y_1 = A \sin(k_1 x - \omega_1 t), \qquad \text{where: } k_1 = \frac{\omega_1}{c_1} \tag{16.1.1}$$

and

$$y_2 = A \sin(k_2 x - \omega_2 t). \qquad k_2 = \frac{\omega_2}{c_2} \tag{16.1.2}$$

The amplitude,Y, of the wave is the sum of the two.individual waves:

$$Y = y_1 + y_2 = A \sin(k_1 x - \omega_1 t) + A \sin(k_2 x - \omega_2 t)$$

Because any analytic function can be written as the sum of its odd and even parts, the wave numbers and frequencies can be re-written as:

$$k_1 = \frac{1}{2}[k_1 + k_2] + \frac{1}{2}[k_1 - k_2],$$

and,

$$k_2 = \frac{1}{2}[k_1 + k_2] - \frac{1}{2}[k_1 - k_2], \tag{16.1.3}$$

and

$$\omega_1 = \frac{1}{2}[\omega_1 + \omega_2] + \frac{1}{2}[\omega_1 - \omega_2]$$

and

$$\omega_2 = \frac{1}{2}[\omega_1 + \omega_2] - \frac{1}{2}[\omega_1 - \omega_2], \tag{16.1.4}$$

Defining the average of the wavenumbers, $k_0$, and frequencies, $\omega_0$, as;

$$k_0 = \frac{k_1 + k_2}{2}, \qquad \text{and } \omega_0 = \frac{\omega_1 + \omega_2}{2}.$$

And using the identity:

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) \qquad 16.1.5$$

We can rewrite the amplitude, Y, due to the interaction of the two waves:

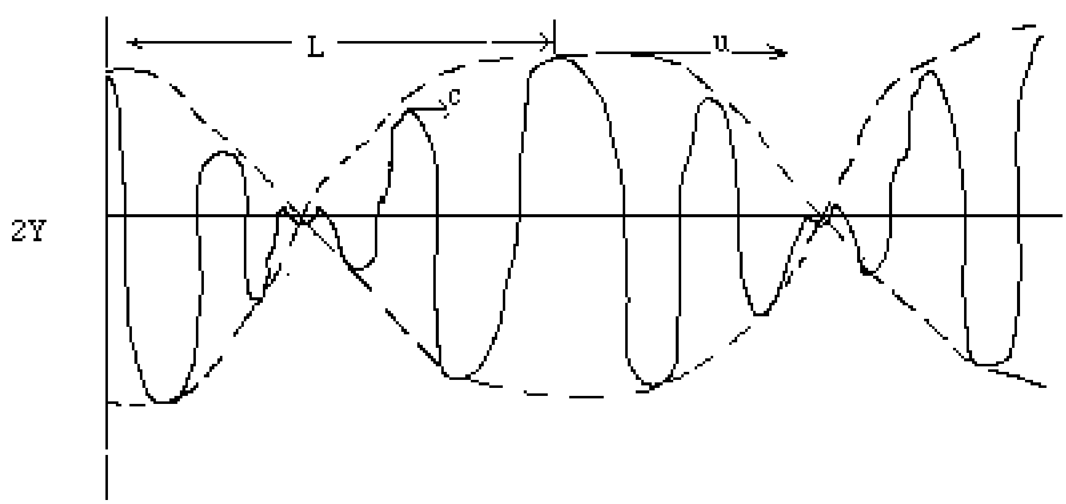
$$Y = y_1 + y_2 = 2A \cos\left[\frac{(k_1 - k_2)}{2}x - \frac{(\omega_1 - \omega_2)}{2}t\right] \bullet \sin\left[\frac{(k_1 + k_2)}{2}x - \frac{(\omega_1 + \omega_2)}{2}t\right] \qquad 16.1.6$$

$$\text{Let } k_1 - k_2 = \Delta k, \ \omega_1 - \omega_2 = \Delta\omega, \qquad 16.1.7$$

then:

$$y_1 - y_2 = 2A \cos \frac{1}{2}(\Delta kx - \Delta\omega t) \ \sin \ (k_o x - \omega_o t) \qquad 16.1.8$$

which represents two modes of wave propagation, a longer period carrier propagating at $\omega_0$ and a higher frequency component of $\Delta\omega$.




Thus the carrier wave propagates at a phase velocity of $c = \frac{\omega_0}{k_0}$ with an angular frequency, $(\omega_0 = 2\pi f_0)$ modulated by a wave with a frequency, $\Delta\omega = 2\pi\Delta f$.. The modulation envelope, or the wave packet, propagates at a group velocity of $u = \Delta\omega/\Delta k$ .. Hence c is different than u for a dispersive wave.

Summarizing the properties of a dispersive media: