Empirical Orthogonal Function (EOF) Analysis (MET5090, Ahlquist, 60 minutes)

(a) Let $\delta_{k,j}$ denote the Kronecker delta function defined as

$$\delta_{k,j} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \tag{1}$$

Let $\{e_k(x)\}$ be any complex-valued basis that is orthonormal with respect to the standard inner product, i. e.,

$$\sum_{x=1}^{M} e_k(x) e_j^*(x) = \delta_{k,j} \tag{2}$$

Because $\{e_k(x)\}$ is a basis, there is a unique way to represent any $f(x,t)$ in the form

$$f(x,t) = \sum_{k=1}^{M} p_k(t) e_k(x). \tag{3}$$

Starting from (3), use the orthonormal relation (2) to show that

$$p_j(t) = \sum_{x=1}^{M} f(x,t) e_j^*(x). \tag{4}$$

(b) Let $E[\quad]$ denote statistical expectation. Suppose that $f(x,t)$ in part (a) is a stochastic function whose statistical properties are stationary in time. Let $p_j(t)$ be given by (4), and denote the mean of $p_j(t)$ as $\mu_j = E[p_j(t)]$. Show that $\mu_j = 0$ if $E[f(x,t)] = 0$ for all x. Hint: Remember that $\{e_k(x)\}$ is not a random variable, but $f(x,t)$ is.

(c) The covariance of two complex-valued random variables is defined as

$$\text{cov}(p_j, p_k) = E[(p_j(t) - \mu_j)(p_k(t) - \mu_k)^*]. \tag{5}$$

Starting from this definition and assuming $E[f(x,t)] = 0$, use earlier results to show that

$$\text{cov}(p_j, p_k) = \sum_{x=1}^{M} \sum_{x'=1}^{M} e_j^*(x) C_{xx'} e_k(x') \quad \text{where} \quad C_{xx'} \equiv E[f(x,t) f^*(x',t)]. \tag{6}$$

Hint: When you multiply two sums, use x as the index for one summation and x' for the other.

(d) Suppose now that $e_1(x), e_2(x), \ldots, e_M(x)$ are EOFs, i. e., they satisfy (2) and

$$\sum_{x'=1}^{M} C_{xx'} e_k(x') = \lambda_k \, e_k(x) \tag{7}$$

where λ_k is the corresponding eigenvalue. Use (6), (7), and (2) to prove that

$$\text{cov}(p_j, p_k) = \lambda_k \, \delta_{k,j} \tag{8}$$

where $\delta_{j,k}$ is given by (1). That is, $\text{cov}(p_j, p_k) = 0$ when $j \neq k$ and $\text{var}(p_j) = \lambda_j$.

(e) Give an example of a situation in which it would be good to do an EOF analysis. What kind of information would you expect to get from the EOF analysis?

(f) Give an example of a problem where traditional EOFs may not be the best tool to analyze an observed function $f(x,t)$. Explain why in detail.

Time Series: Linear Filtering 1 hour Ahlquist

Background: By definition, a linear filter can be written in the form of a convolution integral,

$$y(t) = \int_{-\infty}^{\infty} x(t-t')\, h(t')\, dt'$$

where $x(t)$ is the input signal, $y(t)$ is the output signal, and $h(t)$ is called the weighting function or impulse response function. The Fourier transform of $h(t)$ may be defined as

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-i2\pi ft}\, dt$$

$H(f)$ is called the frequency response function.

(a) Prove the convolution theorem, $Y(f) = X(f) \cdot H(f)$, where $X(f)$ and $Y(f)$ are the Fourier transforms of $x(t)$ and $y(t)$. Hint: Define a new time variable $\tau = t - t'$.

(b) In general, a filter's frequency response function $H(f)$ is complex-valued. It may be written as $H(f) = G(f)\, e^{i\phi(f)}$ where $G(f) \equiv |H(f)|$ is the gain (or amplitude) of $H(f)$ and $\phi(f) = \arctan\{\text{Im}[H(f)]/\text{Re}[H(f)]\}$ is the phase. Explain why it is generally desirable for a filter if $\phi(f) = 0$. That is, what are the consequences in both the frequency and time domains if $\phi(f) \neq 0$?

(c) A filter is called "causal" or "physically realizable" if $h(t) = 0$ for all $t < 0$. Using the convolution integral at the top of this page, explain why this property applies to filters which can work in real time, i.e., which can filter a signal as it happens, as opposed to acausal filters which require the complete time series from beginning to end before the filter can be applied.

(d) Explain why any causal filter introduces a phase shift, i.e., explain why the phase, $\phi(f)$, of $H(f)$ is generally nonzero for a causal filter. Hint: How does the Fourier transform integral for $H(f)$ "simplify" in the case of a causal filter? Then recall that $h(t)$ is real-valued and that $e^{-i2\pi ft} = \cos(2\pi ft) - i\sin(2\pi ft)$.

(e) Suppose you are thinking about applying a filter to a time series. How would you choose the impulse response function $h(t)$ or equivalently the frequency response $H(f)$ for your filter?

(f) Suppose that an impulse response function is given by

$$h(t) = h_{-1}\,\delta(t - \Delta t) + h_0\,\delta(t) + h_1\,\delta(t + \Delta t).$$

$\delta(t)$ is the Dirac delta function defined by $\int_{-\infty}^{\infty} p(t)\,\delta(t)\,dt = p(0)$ for any function $p(t)$. Evaluate the convolution integral at the top of this page to show that

$$y(t) = h_{-1}\,x(t - \Delta t) + h_0\,x(t) + h_1\,x(t + \Delta t))$$

(g) For the filter just specified, compute the frequency response, $H(f)$. Then solve for the filter's gain $G(f)$ and phase $\phi(f)$.

Baroclinic instability (MET 4302/5312, Ahlquist, 60 minutes)

Thoroughly discuss baroclinic instability from the perspective of the two-level linearized quasi-geostrophic equations, where phase speed is given by

$$\begin{aligned} c &= U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} \pm \delta^{1/2} \\ \text{where} \quad \delta &\equiv \frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2} - \frac{U_T^2(2\lambda^2-k^2)}{k^2+2\lambda^2)} \\ U_m &\equiv \frac{1}{2}(U_1+U_3) \\ U_T &\equiv \frac{1}{2}(U_1-U_3) \\ \lambda^2 &\equiv f_0^2/\left[\sigma(\Delta p)^2\right] \end{aligned}$$

and where U_1 and U_3 are upper and lower level mean zonal winds or ocean currents, σ is the static stability (always positive for quasigeostrophic applications), and Δp is the pressure difference between the two layers. In your discussion of baroclinic instability, you should explain:

(a) Are the waves dispersive?

(b) What name is given to λ^{-1}, and what is its significance? (λ^2 has units of m^{-2}).

(c) A growing perturbation in a baroclinically unstable basic state has a 3-D velocity vector that moves air parcels on a slant where the angle is between horizontal and the slope of a potential temperature surface. Explain why this makes sense physically.

(d) What physical effect helps to stabilize "long" waves? What physical effect helps to stabilize "short" waves?

(e) What role does baroclinic instability play in the Earth's atmospheric energy budget?

As part of your answer, you should make a sketch of a stability diagram. Do this by labeling regions of stability and instability on a parameter space diagram showing $k^2/(2\lambda^2)$ on the abscissa (x-axis) and $2\lambda^2 U_T/\beta$ on the ordinate (y-axis).

Black-body radiation 1 hour Ahlquist

(a) What is the formula for the flux density in watts per square meter emitted by a black body at temperature T? What is the name of the constant in this formula?

(b) According to Kirchhoff's law, how are absorptivity and emissivity related for a body in thermodynamic equilibrium?

(c) What is the absorptivity of a black body? What is the emissivity of a black body?

(d) Consider a simple radiative model in which the atmosphere is treated as N isothermal layers, each of which absorbs and emits as a black body at the top and bottom of the layer. Let T_i represent the temperature of the i-th layer. Let T_0 represent the temperature of the surface, which is also nearly a black body in the infrared. Assuming steady state, write the flux balance equation for the i-th atmospheric layer where $i = 1, \ldots, N-1$.

(e) What is the flux balance equation for the N-th layer whose top is at outer space?

Time series: Least squares and EOF analysis (Ahlquist, 1 hour)

Suppose you want to use ensemble forecasting to make a 24-hour forecast of quantity $q(t)$ for day t. Let $f(t,i)$ denote the i-th of N 24-hour forecasts that all apply to day t. We seek an ensemble forecast for $q(t)$ of the form

all 4 parts

$$f(t) = \sum_{i=1}^{N} a_i f(t,i) \tag{1}$$

To allow for bias correction, we can take the N-th forecast to be $f(t,N) = 1$.

(a) Minimize the square error, $E = \sum_t (q(t) - f(t))^2$, to derive the equations that must be solved to determine $a_1, \ldots, a_N$ based on an old set of forecasts and analyses for $q(t)$. (This is often called the training data set.)

(b) A common difficulty with the traditional least squares approach is that some of the predictors, $f(t,i)$, may be strongly correlated with each other. In that case, the equations you derived in part (a) are ill-conditioned, meaning that a small change in a term in the equations for $a_1, \ldots, a_N$ can make a big difference in the solution. For example, consider this example where $N = 2$.

$$\begin{pmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2}$$

I am taking to be minus 1. This can not be solved if $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ In that case a_1 or a_2 must be 0.

Solve for a_1 and a_2 when $\epsilon = 0.01$. Solve again for a_1 and a_2 when $\epsilon = -0.01$.

(c) One way around the ill-conditioning difficulty is to perform an empirical orthogonal function (EOF) analysis on the old set of forecasts. Let $e_j(i)$ denote the i-th element of the j-th EOF. Then, using the old forecasts again, we compute the j-th principal component (PC),

$$p_j(t) = \sum_{i=1}^{N} f(t,i) e_j(i) / \sum_{i=1}^{N} e_j^2(i). \tag{3}$$

Similar to but instead of (1), we write our forecast as

$$f(t) = \sum_{j=1}^{n} b_j p_j(t). \tag{4}$$

where we choose $n < N$ if the EOF analysis shows that forecasts are strongly correlated. Using the same equations that you derived for part (a) and the fact that the PCs are orthogonal, i.e., $\sum_t p_j(t) p_k(t) = 0$ if $j \neq k$, show that the equations for b_j do not suffer from ill-conditioning. In fact, the formula for b_j is completely independent of the formula for any other b_k.

(d) What information is contained in the EOFs, $e_j(i)$, for this application?

Time series analysis (45 minutes) (Ahlquist)

(a) Define the term "spectrum" as used in time series analysis.
(b) Explain in detail how a spectrum can be estimated from a time series observed at discrete times, $t = k\Delta t$. Include a discussion of smoothing, aliasing, leakage, and confidence limits.
(c) Give an example of a problem in meteorology or oceanography where a spectrum could provide useful information. Discuss the example. (The answer to this part of the question should be one or two paragraphs.)

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Dr. Ahlquist: Rossby wave reflection (OCP 5253) (60 minutes)
Consider the Rossby wave dispersion relation

$$\sigma = -\frac{\beta k}{k^2 + l^2 + F}$$

where k and l are the x and y wavenumbers (2π/wavelength) and where β and F are constants.
(a) Let β and F be fixed. Derive and plot the curve in the (k, l) plane which describes the set of Rossby waves which all have the same frequency, σ.
(b) Align Cartesian coordinates so that the x-axis points eastward and the y-axis points northward. Suppose a coastline runs in a straight line, making an angle α with the x-axis, and an ocean occupies the region to the east of the coastline. Consider the problem of oceanic Rossby waves reflecting from the coastline. Write the streamfunction as

$$\begin{aligned} \psi(x,y,t) &= \psi_I(x,y,t) + \psi_R(x,y,t) \\ \text{where} \quad \psi_I(x,y,t) &\equiv A_I e^{i(k_I x + l_I y - \sigma_I t)} \\ \psi_R(x,y,t) &\equiv A_R e^{i(k_R x + l_R y - \sigma_R t)}. \end{aligned}$$

The subscripts I and R refer to the incident and reflected parts of the wave field. Take the boundary condition at the wall to be no flow into or out of the wall. If A, k, l, and s are given, find A_R, k_R, l_R, and σ_R. Show all details.
Hint: Use your answer to part (a) as you carefully apply the boundary conditions.

Dr. Ahlquist: Fundamental dynamics (any basic fluid dynamics course) (60 minutes)
When water flows from a faucet, the radius, r, of the stream of water decreases as the distance, z, from the faucet increases.
a) List as many physical mechanisms as you can that might affect the radius of the stream of water from a faucet.
b) In your opinion, what is the most important reason why r decreases as z increases? Use that physical principle to derive a formula for r as a function of z and any other relevant parameters.
c) Describe how to do an experiment using your kitchen sink and common household items which would test the theory you derived in the previous step.

Basic Dynamics (MET4301/MET5311, Ahlquist, 45 minutes)

a) Dynamics involves the equations of motion, continuity, and thermodynamics. In words, what physical laws are represented by these three equations?

b) Write the three-dimensional equations of motion, continuity, and thermodynamics mathematically for a rotating reference frame. Briefly explain what each term in each equation means.

c) Relative vorticity is defined as $\partial v/\partial x - \partial u/\partial y$, where u and v are the x and y components of velocity. Show mathematically that $\partial v/\partial x$ is the angular velocity of a line of fluid initially parallel to the x-axis. (Hint: Draw a short line segment of fluid initially parallel to the x-axis and consider what happens to it in time when v is not constant in x.) Show mathematically that $-\partial u/\partial y$ is the angular velocity of a line of fluid initially parallel to the y-axis.

d) Explain why the concept of vorticity is necessary in fluid dynamics. That is, why is angular velocity inadequate when describing fluids?

The first question covers material from OCP5253 (Geophysical Fluid Dynamics), but the question, except for part (d), should be within the grasp of anyone who knows the definition of circulation and knows some basics of line integrals and vector fields.

The second question draws on basic physics and fluid dynamics to see how well the student can discuss physical processes, weave a theory, and decide what is necessary to test it. This is the type of question that I would like to emphasize on preliminary exams because it stresses reasoning more than memorization.

1. Kelvin's theorem says that circulation around a curve C is conserved following fluid motion if the fluid is barotropic on C and if friction vanishes on C.

a) Derive Kelvin's theorem from the vector equation of motion.

b) Prove that circulation around a vortex tube is independent of where the curve C circles the vortex tube.

c) Prove that vortex tubes move with the fluid when Kelvin's theorem applies.

d) Explain how the conservation of potential vorticity is really Kelvin's theorem for a specially chosen contour.

2. When water flows from a faucet, the radius, r, of the stream of water decreases as the distance, z, from the faucet increases.

a) List as many physical mechanisms as you can that might affect the radius of the stream of water.

b) In your opinion, what is the most important reason why r decreases as z increases? Use that physical principle to derive a formula for r as a function of z and any other relevant parameters.

c) Describe how to do an experiment using your kitchen sink and common household items which would test the theory you derived in the previous step.

Cloud Physics (Ahlquist)

In as much detail as possible, discuss how cloud droplets grow from roughly a micrometer in size to become a raindrop.